

Complete Convergence for M-Pairwise Negatively Dependent Random Variables

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ABSTRACT

Hsu and Robbins (1947) introduce the concept *complete convergence* as follows. A sequence $\{U_n, n \geq 1\}$ of random variables is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. The converse is true if the $\{U_n, n \geq 1\}$ are independent. They also show that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdős (1949) proved the converse. The result of Hsu-Robbins-Erdős is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. In this paper, let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$ and $\{X, X_n, n \geq 1\}$ be a sequence of m-pairwise negatively dependent random variables. We study the complete convergence for m-pairwise negatively dependent random variables under mild condition $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$. Our results obtained in the paper generalize the corresponding ones for pairwise independent and identically distributed random variables and also pairwise negatively dependent random variables.

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1. Introduction and main results

Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$, and let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise independent and identically distributed random variables. Denote $S_n = \sum_{i=1}^n X_i$ for each $n \geq 1$. Now, we consider the following assumptions:

- (i) $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$
- (ii) $S_n/a_n \rightarrow 0$ a.s.;
- (iii) $\sum_{i=1}^n |X_i|/a_n \rightarrow 0$ a.s.

Sung [1] proved that the three assumptions above are equivalent for pairwise independent and identically distributed random variables. In addition, he presented some results on complete convergence for pairwise independent and identically distributed random variables. Shen, Zhang and Volodin [2] generalized the main results of Sung [1] to the case of pairwise negatively dependent random variables. In this paper, we extend some results due to Shen, Zhang and Volodin [2] concerning complete convergence, in the sense of m –pairwise negatively dependent random variables. Our main results are as follows.

Theorem 1.1. *Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$. Let $\{X, X_n, n \geq 1\}$ be a sequence of m-pairwise negatively dependent with identical distribution. If $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^n [X_i - EX_i 1(|X_i| \leq a_n)]\right| > a_n \varepsilon\right) < \infty. \quad (1)$$

Theorem 1.2. Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow \infty$. Let $\{X, X_n, n \geq 1\}$ be a sequence of m -pairwise negatively dependent random variables with identical distribution. If $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$, then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq k \leq n} |S_k| > a_n \varepsilon\right) < \infty, \quad \forall \varepsilon > 0. \quad (2)$$

Remark 1.3. m -pairwise negative dependence is a very wide dependence structure, which includes independent sequence as a special case. Hence, Theorems 1 and 2 generalize the corresponding ones for pairwise independent and identically distributed random variables and pairwise negatively dependent random variables to the case of m -pairwise negatively dependent random variables.

Here and thereafter, let $1(A)$ be the indicator function of the set A . The symbol C denotes a generic positive constant not depending on n , which may be different for each appearance. Denote $a_0 = 0$, $X^+ = X1(X \geq 0)$, and $X^- = -X1(X < 0)$.

2. Preliminaries

In this section, notation, technical definitions, and lemmas are needed in connection with the main results will be presented.

The concept of negative dependence of random variables was introduced by Lehmann [3]. A collection of random variables $\{X_1, \dots, X_n\}$ is said to be negatively dependent if for all $x_1, \dots, x_n \in R$,

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) \leq P(X_1 \leq x_1) \dots P(X_n \leq x_n) \text{ and} \\ P(X_1 > x_1, \dots, X_n > x_n) \leq P(X_1 > x_1) \dots P(X_n > x_n)$$

A sequence of random variables $\{X_i, i \geq 1\}$ is said to be negatively dependent if for any $n \geq 1$, the collection $\{X_1, \dots, X_n\}$ is negatively dependent. A sequence of random variables $\{X_i, i \geq 1\}$ is said to be pairwise negatively dependent if for all $x, y \in R$ and for all $i \neq j$,

$$P(X_i \leq x, X_j \leq y) \leq P(X_i \leq x)P(X_j \leq y).$$

It is well known and easy to prove that $\{X_i, i \geq 1\}$ is pairwise negatively dependent if and only if for all $x, y \in R$ and for all $i \neq j$,

$$P(X_i > x, X_j > y) \leq P(X_i > x)P(X_j > y)$$

There is a stronger dependence structure so-called the negative association which was introduced by Joag-Dev and Proschan [4]. By Property P3 in Joag-Dev and Proschan [4], negative association implies pairwise negative dependence. For examples about negatively dependent random variables that are not negatively associated, see [4, p. 289]. Of course, pairwise independence implies pairwise negative dependence, but pairwise independence and negative dependence do not imply each other. Joag-Dev and Proschan [4] showed that many natural examples such as negatively correlated normal distribution, permutation distribution, random sampling without replacement are negatively associated with random variables, and therefore they are negatively dependent.

Definition 2.1. (see [5-7]) Let $m \geq 1$ be a fixed integer. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be m -pairwise negatively dependent if for all positive integers j and k with $|j - k| \geq m$, X_j and X_k are negatively dependent.

Clearly, pairwise negative dependence is the special case $m = 1$ of the concept of m -pairwise negative dependence.

The following lemmas come from Lemma of Lehmann [3], see [8] for a more direct proof.

Lemma 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise negatively dependent random variables, let $f_n: R \rightarrow R$ be measurable functions, $n \geq 1$. If the sequence $\{f_n, n \geq 1\}$ consists of only nondecreasing functions or only nonincreasing functions, then $\{f_n(X_n), n \geq 1\}$ is a sequence of pairwise negatively dependent random variables.

By the definition of m -pairwise negatively dependent random variables and Lemma 2.2, we can get the following property for m -pairwise negatively dependent random variables, which can be found in Wang et al. [6].

Lemma 2.3. Let $\{X_n, n \geq 1\}$ be a sequence of m -pairwise negatively dependent random variables, let $f_n: R \rightarrow R$ be measurable functions, $n \geq 1$. If the sequence $\{f_n, n \geq 1\}$ consists of only nondecreasing functions or only nonincreasing functions, then $\{f_n(X_n), n \geq 1\}$ is a sequence of m -pairwise negatively dependent random variables.

The following lemma is a case of Wang et al. [6, Theorem 2.1].

Lemma 2.4. Let $\{X_n, n \geq 1\}$ be a sequence of m -pairwise negatively dependent random variables. Then there exists a positive constant $C_{m,p}$ depending only on m and p such that, for every $n \geq m$,

$$E|\sum_{i=1}^n X_i|^p \leq C_{m,p} \sum_{i=1}^n E(|X_i|^p) \quad \text{for } 1 \leq p \leq 2.$$

The next three lemmas come from Sung [1].

Lemma 2.5. Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$. Then the following properties hold.

- (i) $\{a_n, n \geq 1\}$ is a strictly increasing sequence with $a_n \uparrow \infty$.
- (ii) $\sum_{n=1}^{\infty} P(X > a_n) < \infty$ if and only if $\sum_{n=1}^{\infty} P(X > 2a_n) < \infty$.
- (iii) $\sum_{n=1}^{\infty} P(X > a_n) < \infty$ if and only if $\sum_{n=1}^{\infty} P(X > \alpha a_n) < \infty$ for any $\alpha > 0$.

Lemma 2.6. If $\{a_n, n \geq 1\}$ is a sequence of positive constants with $a_n/n \uparrow$ and X is a random variable, then

$$\frac{n}{a_n} E|X|1(|X| \leq a_n) \leq \sum_{n=0}^{\infty} P(|X| > a_n)$$

Lemma 2.7. Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow \infty$ and X is a random variable. If $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$, then $(n/a_n)E|X|1(|X| \leq a_n) \rightarrow 0$.

3. Proof of main results

The proof of main results is inspired by the proof of Theorem 2 and Theorem 3 of Shen et al. [2] and from Lemma 2.3 and Lemma 2.4.

Proof of Theorem 1.1. Note that the condition $a_n/n \uparrow$ implies that

$$\sum_{n=i}^{\infty} \frac{1}{a_n^2} \leq \sum_{n=i}^{\infty} \frac{i^2}{a_i^2 n^2} \leq \frac{i^2}{a_i^2} \sum_{n=i}^{\infty} \frac{1}{n^2} \leq \frac{i^2}{a_i^2} \cdot \frac{2}{i} = \frac{2i}{a_i^2}. \tag{3}$$

For fixed $n \geq 1$, let

$$Y_i = -a_n 1(X_i < -a_n) + X_i 1(|X_i| \leq a_n) + a_n 1(X_i > a_n).$$

Since $\{X_i, i \geq 1\}$ are m -pairwise negatively dependent, it is well known that $\{Y_i, i \geq 1\}$ are also m -pairwise negatively dependent (by Lemma 2.3).

We now estimate the left hand side of (1) as follows.

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^n [X_i - EX_i 1(|X_i| \leq a_n)]\right| > a_n \varepsilon\right) \\ & \leq \sum_{n=1}^{\infty} n^{-1} P(\cup_{i=1}^n (|X_i| > a_n)) + \sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^n (Y_i - EX_i 1(|X_i| \leq a_n))\right| > a_n \varepsilon\right) \\ & \leq \sum_{n=1}^{\infty} P(|X| > a_n) + \sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^n (Y_i - EY_i)\right| > \frac{a_n \varepsilon}{2}\right) \\ & + \sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^n (EY_i - EX_i 1(|X_i| \leq a_n))\right| > \frac{a_n \varepsilon}{2}\right) \doteq I_1 + I_2 + I_3. \end{aligned}$$

To prove (1), it only needs to be shown that $I_2 < \infty$ and $I_3 < \infty$. Note that $I_1 < \infty$ by the assumption $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$.

Since $\{Y_i - EY_i, 1 \leq i \leq n\}$ are m -pairwise negatively dependent random variables by Lemma 2.3, we have

$$\begin{aligned} I_2 &\leq C \sum_{n=1}^{\infty} n^{-1} a_n^{-2} \sum_{i=1}^n E(Y_i - EY_i)^2 \leq C \sum_{n=1}^{\infty} n^{-1} a_n^{-2} \sum_{i=1}^n EY_i^2 \\ &\leq C \sum_{n=1}^{\infty} a_n^{-2} EX_i^2 1(|X_i| \leq a_n) + C \sum_{n=1}^{\infty} P(|X_i| > a_n) \\ &= C \sum_{n=1}^{\infty} a_n^{-2} EX^2 1(|X| \leq a_n) + C \sum_{n=1}^{\infty} P(|X| > a_n) \\ &\leq C \sum_{n=1}^{\infty} a_n^{-2} EX^2 1(|X| \leq a_n) + C \end{aligned} \tag{4}$$

where we have applied Markov's inequality in the first inequality, Lemma 2.4 in the third inequality and the assumption $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$ in the last step.

Combining (3) with (4), we have

$$\begin{aligned} I_2 &\leq C \sum_{n=1}^{\infty} a_n^{-2} \sum_{i=1}^n EX^2 1(a_{i-1} < |X| \leq a_i) + C = C \sum_{i=1}^{\infty} EX^2 1(a_{i-1} < |X| \leq a_i) \sum_{n=i}^{\infty} a_n^{-2} + C \\ &\leq C \sum_{i=1}^{\infty} EX^2 1(a_{i-1} < |X| \leq a_i) i a_i^{-2} + C \leq C \sum_{i=1}^{\infty} i P(a_{i-1} < |X| \leq a_i) + C \\ &\leq C \sum_{i=0}^{\infty} P(|X| > a_i) + C < \infty. \end{aligned}$$

Next, we prove $I_3 < \infty$. It is easily seen that

$$I_3 \leq \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n P(|X_i| > a_n) > \frac{\epsilon}{2}\right) = \sum_{n=1}^{\infty} n^{-1} P\left(nP(|X| > a_n) > \frac{\epsilon}{2}\right).$$

In the following we prove $nP(|X| > a_n) \rightarrow 0$. Note that $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$ and $0 \leq P(|X| > a_n) \downarrow$ as $n \uparrow$; we have $P(|X| > a_n) = o(1/n)$, which implies that $nP(|X| > a_n) \rightarrow 0$. Hence, $I_3 < \infty$. This completes the proof of the theorem.

Proof of Theorem 1.2. We use the same notations as those in Theorem 1.1. We estimate the left hand side of (2) as follows.

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq k \leq n} |S_k| > a_n \epsilon\right) \\ &\leq \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n P(|X_i| > a_n) + \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k X_i 1(|X_i| \leq a_n) > a_n \epsilon\right) \\ &\leq \sum_{n=1}^{\infty} P(|X| > a_n) + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n |X_i 1(|X_i| \leq a_n)| > a_n \epsilon\right) \\ &= \sum_{n=1}^{\infty} P(|X| > a_n) + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n |X_i 1(|X_i| \leq a_n) - Y_i + Y_i - EX_i 1(|X_i| \leq a_n) + EX_i 1(|X_i| \leq a_n)| > a_n \epsilon\right) \end{aligned}$$

$$\begin{aligned} &\leq C + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n |a_n 1(X_i < -a_n) - a_n 1(X_i > a_n)| \right. \\ &\quad \left. > \frac{a_n \epsilon}{3}\right) + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n |Y_i - EX_i 1(|X_i| \leq a_n)| > \frac{a_n \epsilon}{3}\right) \\ &\quad + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n |EX_i 1(|X_i| \leq a_n)| > \frac{a_n \epsilon}{3}\right) \\ &\leq C + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n 1(|X_i| > a_n) > \frac{\epsilon}{3}\right) \\ &\quad + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n |Y_i - EX_i 1(|X_i| \leq a_n)| > \frac{a_n \epsilon}{3}\right) \\ &\quad + \sum_{n=1}^{\infty} n^{-1} P\left(nE|X|1(|X| \leq a_n) > \frac{a_n \epsilon}{3}\right) \doteq C + J_1 + J_2 + J_3. \end{aligned}$$

To prove (2), it remains to show $J_i < \infty$ for $i = 1, 2, 3$. By Markov's inequality and the assumption $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$, we have

$$J_1 \leq C \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n P(|X_i| > a_n) = C \sum_{n=1}^{\infty} P(|X| > a_n) < \infty$$

By the assumptions of Theorem 1.2 and Lemma 2.7, we have $(n/a_n)E|X|1(|X| \leq a_n) \rightarrow 0$, which implies that $J_3 < \infty$.

In the following, we will prove $J_2 < \infty$. It is easily checked that

$$\begin{aligned} J_2 &\leq \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n |Y_i - EY_i| > \frac{a_n \epsilon}{6}\right) + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n \right. \\ &\quad \left. | -P(X_i < -a_n) + P(X_i > -a_n) | > \frac{\epsilon}{6}\right) \\ &\leq \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n (Y_i - EY_i)^+ > \frac{a_n \epsilon}{12}\right) \\ &\quad + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^n (Y_i - EY_i)^- > \frac{a_n \epsilon}{12}\right) + \sum_{n=1}^{\infty} n^{-1} P\left(nP(|X| > a_n) > \frac{\epsilon}{6}\right) \\ &\doteq J_{21} + J_{22} + J_{23}. \end{aligned}$$

Note that, for fixed $n \geq 1$, $\{(Y_i - EY_i)^+, 1 \leq i \leq n\}$ and $\{(Y_i - EY_i)^-, 1 \leq i \leq n\}$ are both m -pairwise negatively dependent random variables by Lemma 2.3. Hence, similar to the proof of $I_2 < \infty$ in Theorem 1.1, we have

$$J_{21} \leq C \sum_{n=1}^{\infty} n^{-1} a_n^{-2} \sum_{i=1}^n E[(Y_i - EY_i)^+]^2 \leq C \sum_{n=1}^{\infty} a_n^{-2} EY_1^2 < \infty$$

Similarly, we have $J_{22} < \infty$. Similar to the proof of $I_3 < \infty$ in Theorem 1.1, we can get that $J_{23} < \infty$. Therefore, $J_2 < \infty$ follows by the statements above. This completes the proof of the theorem.

4. Conclusions

The paper proves a theorem on the complete convergence for m -pairwise negatively dependent random variables under mild condition $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$. The concepts of negative association and pairwise negative dependence can all be extended to random vectors in Hilbert spaces. It is interesting to see if the results in this note can be extended to dependent random vectors taking values in Hilbert spaces. We state the problem here for future research.

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