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Asymmetric Buckling of the Annular Plate with an Internal Source of Heat

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ABSTRACT

In this article, the axisymmetric stability of an elastic annular plate with distributed internal heat source under uniform constant compressive radial loading at the outer edge is investigated. The governing and compatibility equations of the annular plate are taken from modified Karman's equations, which are solved by the shooting method. The effect of the density of distributed internal heat source on critical load is investigated and discussed in detail. The numerical results of the plate without an internal heat source are in good agreement with the results from the literature, validating the effectiveness and advantages of the current method. The critical buckling load increases with a rise of the density of distributed internal heat sources. Moreover, an increase of wave number increases the critical buckling loads of the plate. The results of this research can be considered as benchmark solutions for buckling of the annular plates with an internal heat source.

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1. Introduction

The axial asymmetric stability problem of circular plates subjected to uniformly distributed loads was first solved by Bryan [1]. The variational method has been used to give equilibrium equations for the deflection and boundary conditions, from which the critical load for the plate was found. The buckling of a circular annular plate subjected to shearing forces distributed along the edges was studied by Dean in 1924 [2]. The axisymmetric buckling of a thin annular plate subjected to radial compressive force has been studied by Yamaki [3]. The equilibrium equations are integrated in terms of Bessel functions and the constants of integration are determined by using boundary conditions. Majumdar [4] gave a comparison between the theoretical and experimental results. The buckling of the orthotropic annular plate of continuously variable thickness was investigated by Ciancio et al. [5]. Here the analytical solution was found by using the optimized Rayleigh-Ritz method. Wang et al. [6] studied the stability of the round plate on a part of the elastic foundation. The relationship between the number of waves in the deflection function and the hardness of the elastic base was investigated. An exact solution for the thermal buckling of the functionally graded annular plate under thermal loading was found by Khosravi et al. [7] based on Karman's plate theory. Pawlus et al. [8] used the finite difference method and finite element method to study the buckling of the sandwich annular plate with different damaged laminated facing under static and dynamic temperature loads. Based on higher-order shear deformation theory, Zhang et al. [9] presented the free vibration and buckling analyses of the functionally graded graphene platelets reinforced composite annular plate under thermal and mechanical loadings.

Zubov [10] proposed a modification of Von Kármán equations for flexible elastic plates with the distributed dislocations and disclinations or internal sources of heat. The exact solution of a thin circular plate containing distributed disclinations was found in this article. Based on the finite difference method, asymmetric buckling of the circular plate with distributed disclinations was investigated by Zubov et al. [11].

It can be seen that in the above-mentioned reports, there is no investigation about the asymmetric buckling of an elastic annular plate with distributed internal heat source in the literature. Thus, that makes us get the motivation to study the asymmetric buckling of an annular plate containing distributed internal source of heat under uniform constant compressive radial loading at the outer edge. The boundary conditions of an annular plate are C-C (clamp at the outer and inner edge) and S-S (simply supported at the outer and inner edge). Modified Karman's equations have been used for the investigated influence of the density of internal heat source on the Airy stress function, hence the buckling differential equation of an annular plate was proposed. The buckling load has been found by using the shooting method. The critical buckling load increase with the increasing density of the internal heat source.

2. The influence of internal heat source density on airy stress function

Consider the thin elastic circular annular plate with outer radius a, inner radius b and constant thickness h, having internal stresses, caused by a uniformly distributed heat source, under uniform compressive in-plane force at outer edge $N_{ru} = \sigma_0 h$ and inner edge $N_{ri} = \sigma_0 h$ (Figure 1).

The modified Karman equation for plate has the form [10]

$$\Delta\Delta F + \frac{1}{2}Eh\left[\left(\Delta W\right)^2 - \mathrm{tr}\left(\nabla\nabla W \cdot \nabla\nabla W\right)\right] = -Eh\frac{lQ}{\kappa} \tag{1}$$

$$D\Delta\Delta W + \mathrm{tr}\left(\nabla\nabla F \cdot \nabla\nabla W\right) - \Delta F \Delta W = p \tag{2}$$

where W is deflection, F is Airy stress function, E is Young's modulus, ν is a Poisson ratio, Δ is Laplace operator, tr is a trace of tensor, Q is the density of internal heat source, l is linear expansion coefficient and κ is heat transfer coefficient. D is flexural rigidity of the plate, which is defined as follows

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}$$

Suppose that the transverse load p is zero, and the forces applied to the plate boundary lie in its plane. The Karman's equation (2) has a solution $W = 0$, which corresponds to a plane stress state. This state is described by equation (2), and equation (1) takes the form

$$\Delta\Delta F = -Eh\frac{lQ}{\kappa} \tag{3}$$

Assuming that the internal heat source is distributed uniformly, the Airy stress function is symmetric $F = F(r)$.

In polar system coordinates, the equation (3) becomes

$$\frac{\partial^4 F}{\partial r^4} + \frac{2}{r}\frac{\partial^3 F}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^3}\frac{\partial F}{\partial r} = -Eh\frac{lQ}{\kappa} \tag{4}$$

The boundary conditions of the Airy stress function when the load $N_{ru} = \sigma_0 h$ action at the outer edge are presented as

$$\begin{cases} F\big|_{r=a} = 0 \\ \left.\dfrac{\partial F}{\partial r}\right|_{r=a} = -a\sigma_0 \end{cases}; \quad \begin{cases} \left.\dfrac{\partial F}{\partial r}\right|_{r=b} = 0 \\ \left.\left(r\dfrac{\partial^3 F}{\partial r^3} + \dfrac{\partial^2 F}{\partial r^2}\right)\right|_{r=b} = 0 \end{cases} \tag{5}$$

In the case both inner and outer edges are subjected by the same load $N_{ru} = N_{ri} = h\sigma_0$, the boundary conditions of the Airy stress function are presented as follows

$$\begin{cases} F\big|_{r=a} = 0 \\ \left.\dfrac{\partial F}{\partial r}\right|_{r=a} = -a\sigma_0 \end{cases} ; \quad \begin{cases} \left.\dfrac{\partial F}{\partial r}\right|_{r=b} = -b\sigma_0 \\ \left.\left(r\dfrac{\partial^3 F}{\partial r^3} + \dfrac{\partial^2 F}{\partial r^2}\right)\right|_{r=b} = 0 \end{cases} \tag{6}$$

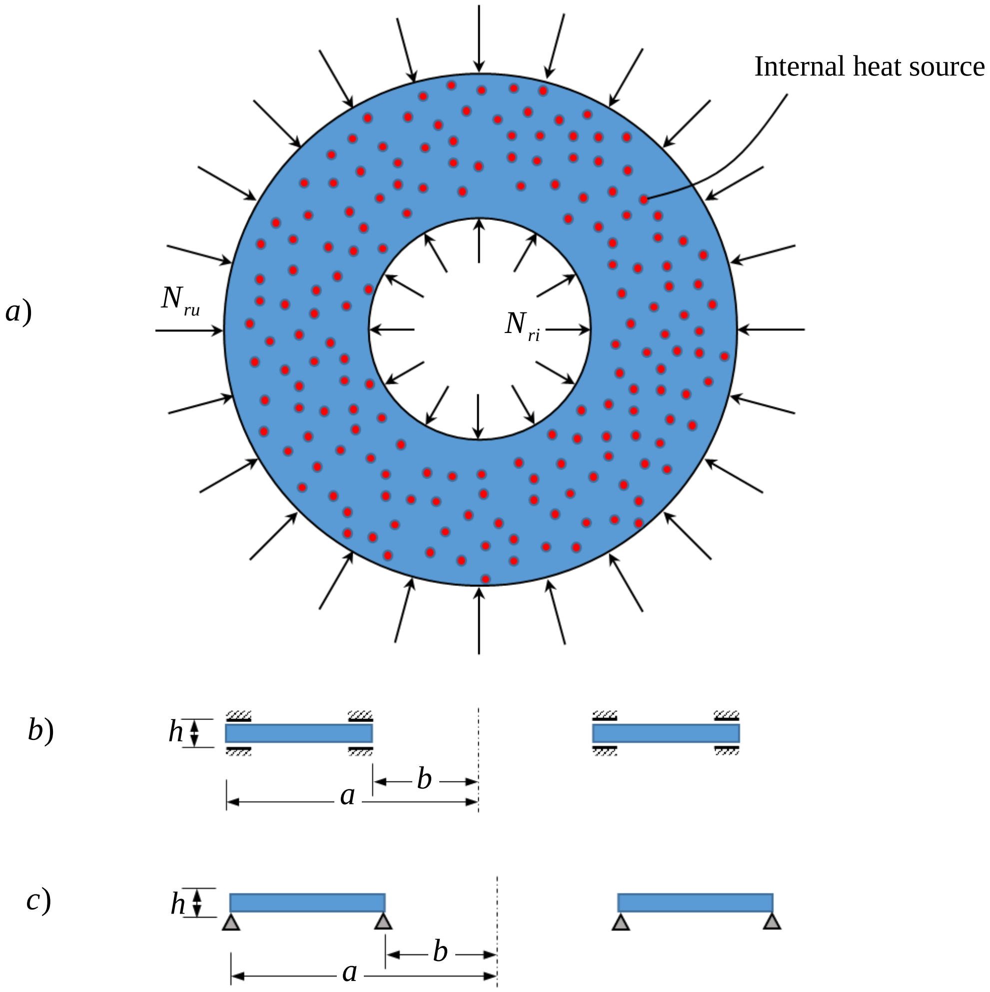

Figure 1. a) *The annular plate with distributed internal heat source subjected inner and outer radial compression. b) The C-C annular plate. c) The S-S annular plate.*

The solution of equation (4) has the form

$$F(r) = -\frac{EhlQ}{64k} r^4 + C_1 + C_2 \ln r + r^2 (C_3 + C_4 \ln r) \tag{7}$$

where C_1, C_2, C_3, C_4 are arbitrary constants. Applying the boundary conditions (5) we found

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$$\begin{cases} C_1 = \dfrac{a^2}{2\left(a^2-b^2\right)}\left[\dfrac{EhlQ}{32\kappa}\left(16b^4\ln a\ln b - 16b^4\left(\ln a\right)^2 - 8b^4\ln b + \right.\right. \\ \left.\left. +4a^2b^2\ln a + 4b^4\ln a - a^4 + 3a^2b^2 - 2b^4\right) + \left(a^2 - 2b^2\ln a\right)\sigma_0\right] \\ C_2 = \dfrac{a^2b^2}{a^2-b^2}\left[\dfrac{EhlQ}{16\kappa}\left(-4b^2\ln b + 4b^2\ln a - a^2 + b^2\right) + \sigma_0\right] \\ C_3 = \dfrac{\dfrac{EhlQ}{16\kappa}\left(-4b^4\ln b + 4a^2b^2\ln a - a^4 + 2a^2b^2 + b^4\right) - a^2\sigma_0}{a^2-b^2} \\ C_4 = \dfrac{EhlQb^2}{8\kappa} \end{cases} \tag{8}$$

Substituting C_1, C_2, C_3 and C_4 into Eq. (7), we get a solution for Airy stress function $F(r)$. Then the components of the stress tensor are defined as

$$\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial F}{\partial r} = -\frac{EhlQ}{16k}r^2 + \frac{C_2}{r^2} + 2C_3 + C_4\left(2\ln r + 1\right) \\ \sigma_\varphi &= \frac{\partial^2 F}{\partial r^2} = -\frac{3EhlQ}{16k}r^2 - \frac{C_2}{r^2} + 2C_3 + \frac{2C_4}{r} \\ \tau_{r\varphi} &= 0 \end{aligned} \tag{9}$$

Let us consider the non-axisymmetric buckling of the annular plate, taking the solution:

$$W\left(r,\varphi\right) = w\left(r\right)\cos m\varphi\ ,\ m = 1,2\ldots \tag{10}$$

where m is number of waves.

The linearized equilibrium equation (2) in polar system coordinates by virtue of (10) is reformed as

$$Dw^{IV} + \frac{2D}{r}w''' + \left(-D\frac{1+2m^2}{r^2} - \sigma_r\right)w'' + \left(D\frac{1+2m^2}{r^3} - \frac{\sigma_\varphi}{r}\right)w' + \left(D\frac{m^4-4m^2}{r^4} - \frac{\sigma_\varphi}{r^2}\right)w = 0. \tag{11}$$

Suppose the plate is clamped at the inner and outer edge, the boundary conditions (BCs) for the deflection w are defined as following

$$\begin{cases} w\big|_{r=a} = 0 \\ \dfrac{\partial w}{\partial r}\bigg|_{r=a} = 0 \end{cases};\ \begin{cases} w\big|_{r=b} = 0 \\ \dfrac{\partial w}{\partial r}\bigg|_{r=b} = 0 \end{cases} \tag{12}$$

The boundary conditions of the plate with simply supported at the inner and outer edges are expressed as

$$\begin{cases} w\big|_{r=a} = 0 \\ \left(\dfrac{\partial w}{\partial r} + \dfrac{\nu}{r}\dfrac{\partial^2 w}{\partial r^2}\right)\bigg|_{r=a} = 0 \end{cases};\ \begin{cases} w\big|_{r=b} = 0 \\ \left(\dfrac{\partial w}{\partial r} + \dfrac{\nu}{r}\dfrac{\partial^2 w}{\partial r^2}\right)\bigg|_{r=b} = 0 \end{cases} \tag{13}$$

3. The Shooting method

Converting 4-th order differential equation to a system of first-order differential equations by replacing:

$$y_1 = w; y_2 = w'; y_3 = w''; y_4 = w''' \tag{14}$$

where index "," stands for derivative of r.

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Then the equivalent system of four first-order differential equations can be written as

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 \\ y_4' = f\left(r, y_1, y_2, y_3, y_4\right) \end{cases} \tag{15}$$

where

$$f = \left(\frac{-m^4 + 4m^2}{r^4} + \frac{\sigma_\varphi}{Dr^2}\right)y_1 + \left(-\frac{1+2m^2}{r^3} + \frac{\sigma_\varphi}{Dr}\right)y_2 + \left(\frac{1+2m^2}{r^2} + \frac{\sigma_r}{D}\right)y_3 - \frac{2}{r}y_4$$

Substituting Eq. (14) into Eq. (12), the boundary conditions are reformed as

$$\begin{cases} y_1(a) = 0 \\ y_2(a) = 0 \end{cases};\ \begin{cases} y_1(b) = 0 \\ y_2(b) = 0 \end{cases} \tag{16}$$

The solution of the above boundary value problem (15), (16) takes the form

$$\mathbf{y}(r) = c_1\mathbf{Y}^1(r) + c_2\mathbf{Y}^2(r) \tag{17}$$

where c_1, c_2 are missing boundary condition at $r = a$ ($c_1 = y_3(a)$; $c_2 = y_4(a)$); $\mathbf{Y}^1$ and $\mathbf{Y}^2$ are solutions of initial boundary value problem, which are expressed as

$$\begin{cases} \mathbf{Y}^{1\prime} = \mathbf{A}(r)\cdot\mathbf{Y}^1; \mathbf{Y}^1(a) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{Y}^{2\prime} = \mathbf{A}(r)\cdot\mathbf{Y}^2; \mathbf{Y}^2(a) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{cases} \tag{18}$$

Initial value problem (15) solved by using Euler's method: $\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{f}\left(r_i, \mathbf{y}_i\right)$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}; \mathbf{f} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ f \end{pmatrix}; h = \frac{b-a}{N}$$

Substituting Eqs. (18) into Eq. (17), we got

$$\begin{cases} y_1(b) = c_1 Y_1^1(b) + c_2 Y_1^2(b) = 0 \\ y_2(b) = c_1 Y_2^1(b) + c_2 Y_2^2(b) = 0 \end{cases} \tag{19}$$

The system of equations (19) is linear. The non-concurrent condition of zero of c_1 and c_2 leads to the following expression

$$\begin{vmatrix} Y_1^1(b) & Y_1^2(b) \\ Y_2^1(b) & Y_2^2(b) \end{vmatrix} = 0 \tag{20}$$

4. Numerical results

Let us consider an annular plate with inner radius a = 0.1, outer radius b = 1 and the thickness h = 0.1. The elastic modulus and the Poisson's ratio of material of the plate are E = 1 and ν = 0.3, respectively. Firstly, the critical buckling load of an elastic annular plate without an internal heat source (Q = 0) under radial compressive load at inner and outer edges is listed in Table 1. As observed, it can be seen that the non-dimensional critical buckling load matches very well with those given by the reference solutions [12]. Besides, comparison results in Table 1 indicate the accord and accuracy of the proposed method.

Table 1. *The non-dimensional critical buckling load σ_0 of elastic plate subjected to radial compressive loads (b/a = 0.1; a/h =10; Q = 0).*

BCs		*m*		
		0	1	2
C-C	Ref [12]	-	0.0043	0.0042
	Present	-	0.0042	0.0041
S-S	Ref [12]	0.0016	0.0016	-
	Present	0.0014	0.0015	-

Next, some novel results of the critical load σ_0 of the annular plate under radial compressive load at the outer edge with the various density of the internal heat source are also shown. The influence of the density of the internal heat source on the non-dimensional critical load of the C-C annular plate is presented in Figure 2. In the case plate is simply supported at the inner and outer edge, the boundary value problem (11) with boundary conditions (13) is solved by the same method. The dimensionless critical buckling load of the S-S annular plate is shown in Figure 3. It can be seen that from Figure 2 and Figure 3, the non-dimensional critical load of an elastic annular plate increases with a rise in the density of the internal heat source. Furthermore, the intersection of critical curves with the abscissa axis in Figure 2 – Figure 3 means that plate buckling is possible without applying an external load, only due to the internal stresses caused by the distributed internal heat source.

Table 2. *The non-dimensional critical buckling load σ_0 of elastic plate with various radius-to-thickness ratios ($Q = 10^6$ W/m; b/a =10; $N_{ru} = \sigma_0 h$; $N_{ri} = 0$).*

BCs	*m*	*a/h*				
		10	20	30	40	50
S-S	0	0.002175	0.000464	0.000217	0.000130	0.000088
	1	0.002141	0.000441	0.000199	0.000117	0.000078
	2	0.002648	0.000429	0.000160	0.000081	0.000048
C-C	0	0.005614	0.00104	0.000463	0.000277	0.000189
	1	0.005017	0.000932	0.000412	0.000245	0.000166
	2	0.004777	0.000835	0.000347	0.000195	0.000127

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Table 3. *The non-dimensional critical buckling load σ_0 of elastic plate with various diameter ratios ($Q = -10^6$ W/m; a/h =10; $N_{ru} = \sigma_0 h$; $N_{ri} = 0$).*

BCs	*m*	*b/a*				
		0.1	0.2	0.3	0.4	0.5
S-S	0	0.000981	0.001249	0.001744	0.002481	0.003644
	1	0.001012	0.00139	0.001903	0.002637	0.00379
	2	0.001893	0.002085	0.00248	0.003139	0.004239
C-C	0	0.000189	0.005077	0.007002	0.009836	0.014388
	1	0.00328	0.004767	0.006753	0.009628	0.014212
	2	0.003342	0.004484	0.006496	0.009893	0.015941

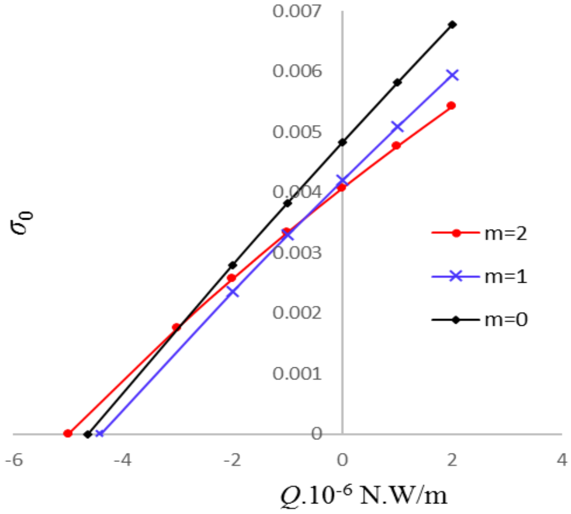

Figure 2. *Influence of density of internal heat source Q on the critical load σ_0 of the C-C plate under axial compressive load at the outer edge (b/a = 0.1; a/h =10; $N_{ru} = \sigma_0 h$; $N_{ri} = 0$).*

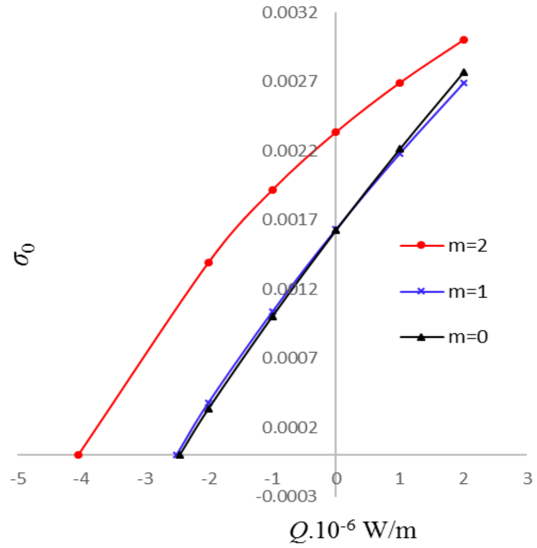

Figure 3. *Influence of density of internal heat source Q on the critical load σ_0 of the S-S plate subjected radial compressive load at the outer edge (b/a = 0.1; a/h =10; $N_{ru} = \sigma_0 h$; $N_{ri} = 0$).*

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Besides, when both inner and outer edges are subjected to the same load $N_{ru} = N_{ri} = \sigma_0 h$, the critical buckling load σ_0 of the C-C and S-S plates with internal heat source is presented in Figure 4 and Figure 5, respectively.

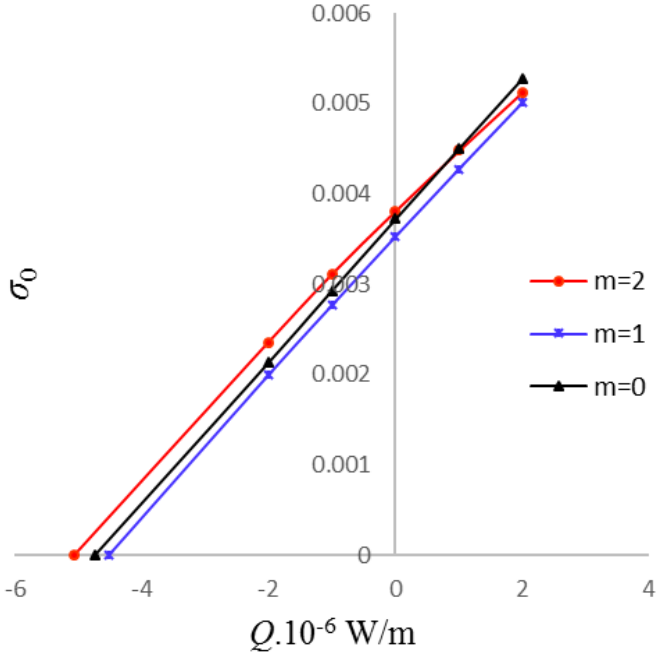

Figure 4. *Influence of density of internal heat source Q on the critical load σ_0 of the C-C annular plate subjected radial compressive load at the outer and inner edges (b/a = 0.1; a/h =10; $N_{ru} = N_{ri} = \sigma_0 h$).*

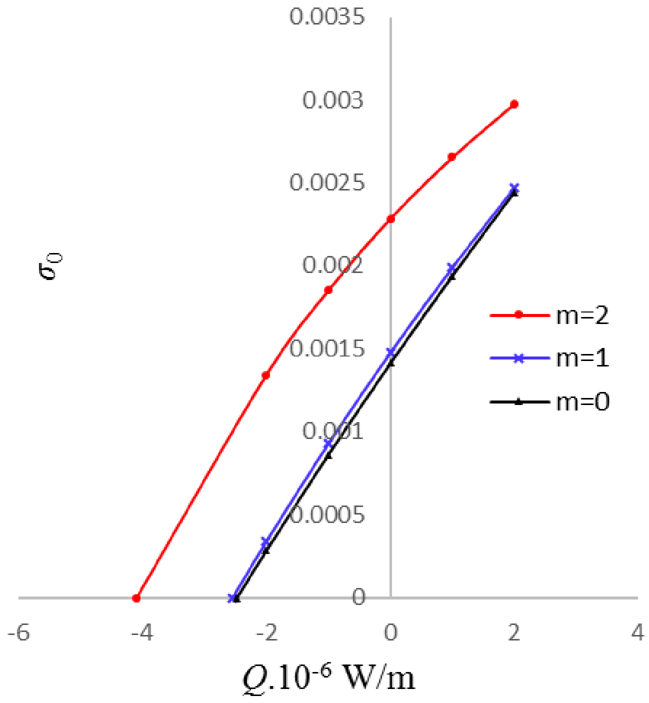

Figure 5. *Influence of density of internal heat source Q on the critical load σ_0 of the S-S plate under axial compressive load at the outer and inner edges (b/a = 0.1; a/h =10; $N_{ru} = N_{ri} = \sigma_0 h$).*

The effect of radius-to-thickness ratios (a/h) on the critical buckling load of the elastic plate with $Q = 10^6$ W/m is shown in Table 2. Finally, the critical buckling load of the SS and CC plates with various diameter ratios (b/a) is presented in Table 3. The density of the internal heat source is taken by $Q = -10^6$ W/m. According to Table 2 and Table 3, the non-dimensional critical buckling load decreases and increases with an increase in the radius-to-thickness ratios and the diameter ratios, respectively.

5. Conclusions

This article deals with investigating the buckling analysis of the elastic annular plate with an internal heat source under radial compressive load based on the Shooting method. The numerical results of the plate without the internal heat source are in good agreement with the results from the literature, validating the effectiveness and advantages of the current method. The effect of the density of the internal heat source and geometries on the non-dimensional critical buckling load has been studied. The buckling load increase with a rise of the density of the internal heat source. In addition, an increase of the radius-to-thickness ratio leads to a decrease of the critical buckling load of the elastic plate. In the case of missing external load, the plate is buckled caused by the internal source of heat. Finally, the numerical results of this research can be considered a benchmark solution for the buckling of elastic plates with the internal heat source.

Acknowledgments

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