DIMENSION RESULT FOR THE POLYNOMIAL ALGEBRA OF SIX VARIABLES AS A MODULE OVER STEENROD ALGEBRA IN SOME DEGREES

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ABSTRACT

Let P_k be the graded polynomial algebra $F_2[x_1, x_2, ..., x_k]$ with the degree of each generator x_i being 1, where F_2 denote the prime field of two elements. We study the hit problem set up by Frank Peterson of finding a minimal set of generators for the polynomial algebra P_k as a module over the mod-2 Steenrod algebra, A. If we consider F_2 as a trivial A-module, then the hit problem is equivalent to the problem of finding a basis of F_2-graded vector space $F_2 \otimes_A P_k$. The problem is still open for $k \ge 5$. It is known that the hit problem is reduced to the case of the degree u of the form $u = s(2^d - 1) + 2^d m$, where s, d, m are non-negative integers such that $\mu(m) < s \le k$. Here, $\mu(m)$ is the smallest number t for which it is possible to write $m = \sum_{1 \le i \le t} (2^{n_i} - 1)$, where $n_i > 0$. In this paper, we study the hit problem of the degree $n = 11.2^{r+2} - 5$ in P_6 for any integer $r > 4$.

Keywords: *Steenrod squares; polynomial algebra; hit problem; algebraic transfer; Steenrod algebra.*

1. INTRODUCTION

Let E_k be an elementary abelian 2-group of rank k. Then,

$$P_k = H^*(E_k) \cong F_2[x_1, x_2, ..., x_k] \quad (1)$$

a graded polynomial algebra in k variables $x_1, x_2, ..., x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field F_2 of two elements.

Being the cohomology of a group, P_k is a module over the mod-2 Steenrod algebra, A. The action of A on P_k is determined by the elementary properties of the Steenrod squares Sq^i and the Cartan formula (see Steenrod and Epstein [1]).

One of our favourite problems is finding the minimal generating set of the polynomial algebra P_k as a module over the Steenrod algebra. That is so-called *hit problem* for the polynomial algebra. If we consider F_2 as a trivial A-module, then the hit problem is equivalent to the problem of finding a basis of F_2-graded vector space

$$F_2 \otimes_A P_k \cong P_k / A^+ P_k \quad (2)$$

where A^+ is an ideal of A generated by all Steenrod squares of positive degrees.

This problem has been first studied by Peterson [2], Singer [3], Wood [4], Priddy [5]... who pointed out its relationship with some classical problems in homotopy theory such as the cobordism theory of manifolds, the modular representation theory of linear groups, Adams spectral sequences of stable homotopy of spheres, and stable homotopy type of the classifying space of finite groups.

The hit problem was studied by many authors (see Boardman [6], Hung [7], Phuc [8], Sum [9-13], Sum-Tin [14; 15], Tin [16-19] and others).

In 1987, Peterson conjectured that, as a module over Steenrod algebra, the polynomial algebra P_k is generated by the monomials of degree n satisfying $\alpha(n+k)\le k$, where $\alpha(n)$ denotes the number of ones in dyadic expansion of a non-negative integer n and proved it for $k\le 2$. The conjecture was established in general by Wood [4]. This is an important tool for determining A-generators for P_k. This result has been further developed by Singer and Silverman.

Let GL_k be the general linear group of rank k over the field F_2. This group acts naturally on P_k by matrix substitution. Since the two actions of A and GL_k upon P_k commute with each other, there is an inherited action of GL_k on $F_2\otimes_A P_k$.

One of the main tools in the study of the hit problem is Kameko's squaring operation

$$Sq_*^0 : F_2\otimes_A P_k \to F_2\otimes_A P_k \qquad (3)$$

This homomorphism is induced by the F_2-linear map, also denoted by $\phi_k : P_k \to P_k$ given by

$$\phi_k(x)=\begin{cases} y, & \text{if } x=\prod_{i=1}^{k} x_i.y^2 \\ 0, & \text{otherwise}\end{cases} \qquad (4)$$

for any monomial $x\in P_k$. Kameko proved the following theorem.

Theorem 1 (Kameko [20]). Let m be a non-negative integer. If $\mu(2m+k)=k$ then

$$\left(Sq_*^0\right)_{(k,m)} : \left(F_2\otimes_A P_k\right)_{2m+k} \to \left(F_2\otimes_A P_k\right)_m, \quad (5)$$

is an isomorphism of GL_k-modules, where

$$\mu(n)=\min\left\{m\in\mathbb{N} : \alpha(n+m)\le m\right\} \qquad (6)$$

By combining this result with the one of Wood, the hit problem is reduced to the case of degree n of the form $n=s\left(2^d-1\right)+2^d m$, where $s,\ d,\ m$ are non-negative integers such that $\mu(m)<\mathrm{s}\le k$.

The computation of the vector space $F_2\otimes_A P_k$ is a hard work, hence one usually consider the dimension of this vector space. Carlisle-Wood proved that the dimension of the vector space $\left(F_2\otimes_A P_k\right)_n$ is bounded by a number that is depended on k.

Moreover, after explicitly determining $F_2\otimes_A P_4$ by the method of Kameko, N. Sum [13] has established a inductive formula by k for the dimension of the vector space $\left(F_2\otimes_A P_k\right)_n$ where n is of general degree.

Theorem 2 (Sum [13]).

Let $n=(k-1)\left(2^d-1\right)+2^d m$ with $d,\ m$ positive integers such that:

$$\begin{cases} 1\le k-3\le \mu(m)\le k-2 \\ \alpha\left(m+\mu(m)\right)=\mu(m)\end{cases} \qquad (7)$$

If $d\ge k-1$ then

$$\dim\left(F_2\otimes_A P_k\right)_n=\left(2^k-1\right)\dim\left(F_2\otimes_A P_{k-1}\right)_m.$$

By a direct computation we see that for d a arbitrary non-negative integer, then there exists a non-negative integer t such that $\mu\left(k(2^s-1)+2^s.d\right)=k$ for every s > t.

Hence, Theorem 1 implies that

$$\left(Sq_*^0\right)^{s-t} : \left(\mathrm{Q}P_k\right)_{k(2^s-1)+2^s.d} \to \left(\mathrm{Q}P_k\right)_{k(2^t-1)+2^t.d} \quad (8)$$

is an isomorphism of GL_k-modules, for every $s\ge t$ and $\mathrm{QP}_k := \mathrm{F}_2\otimes_A P_k$.

N.H.V. Hung has been proved this result is true for $t=k-2$ and for all d.

Tin-Sum [15] extended this result of Hung [7] on the isomorphism of Kameko's homomorphism as follows:

Theorem 3 (Tin-Sum [15]). Let d be an arbitrary non-negative integer. Then

$$\left(Sq_*^0\right)^{s-t} : \left(QP_k\right)_{k\left(2^s-1\right)+2^s d} \to \left(QP_k\right)_{k\left(2^t-1\right)+2^t d} \quad (9)$$

is an isomorphism of GL_k-modules for every $s \geq t$ if and only if $t \geq t(k,d)$.

Here

$$t(k,d) = \max\left\{0,\ k-\alpha(d+k)-\varsigma(d+k)\right\} \quad (10)$$

where $\varsigma(n)$ is the greatest integer u such that n is divisible by 2^u, that means $n = 2^{\varsigma(n)}m$, with m an odd integer.

This result is very significant for calculating the hit problem and the GL_k-invariant of polynomial algebra.

For now on, the tensor product $F_2 \otimes_A P_k$ was explicitly calculated by Peterson [4] for $k = 1;2$, by Kameko [20] for $k = 3$ and by Sum [9] for $k = 4$. However, the problem is still unsolved with $k \geq 5$, even in the case of $k = 5$ with the help of computers.

The main result of the paper is the following.

Main Theorem. Let $n = 11.2^{r+2} - 5$, with r an arbitrary positive integer. Then, there exist(s) exactly 134190 admissible monomials of degree n in P_6, for $r > 4$. Consequently, $\dim\left(F_2 \otimes_A P_6\right)_n = 134190$.

In Section 2, we recall some needed information on the admissible monomials and hit monomials in P_k. The proof of Main Theorem is presented in Section 3.

2. PRELIMINARIES

We first recall some results from Kameko [3], which will be used in the next section.

Notation 2.1. We denote $\mathbb{N}_k = \{1,2,...,k\}$ and $X_{\Im} = \prod_{j \in \mathbb{N}_k \setminus \Im} x_j$, $\Im = \{j_1, j_2, ..., j_s\} \subset \mathbb{N}_k$

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means

$$a = \alpha_0(a).2^0 + \alpha_1(a).2^1 + \alpha_2(a).2^2 + ... \quad (11)$$

for $\alpha_i(a) \in F_2$ and $i \geq 0$.

Let $x = x_1^{a_1} x_2^{a_2} ... x_k^{a_k} \in P_k$. Denote $\tau_j(x) = a_j$, with $1 \leq j \leq k$. Set $\Im_t(x) = \left\{ j \in \mathbb{N}_k : \alpha_t\left(\tau_j(x)\right) = 0 \right\}$ for $t \geq 0$.

Then, we have $x = \prod_{t \geq 0} X_{\Im_t(x)}^{2^t}$.

Following Kameko [20], we define two sequences associated with x by

$$\begin{aligned} \omega(x) &= \left(\omega_1(x), \omega_2(x), \omega_3(x), ...\right) \\ \sigma(x) &= \left(\tau_1(x), \tau_2(x), ..., \tau_k(x)\right) \end{aligned} \quad (12)$$

where

$$\omega_i(x) = \sum_{j=1}^{k} \alpha_{i-1}\left(\tau_j(x)\right) = \deg X_{\Im_{i-1}(x)},\ i \geq 1. \quad (13)$$

The sequence $\omega(x)$ is called the weight vector of x. The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

Definition 2.2. Let x, y be the monomials in P_k. We say that $x < y$ if and only if one of the following holds:

(i) $\omega(x) < \omega(y)$.

(ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Here, the order on the set of sequences of nonnegative integers is the lexicographical one.

Let f, g be homogeneous polynomials of the same degree in P_k. We denote $f \equiv g$ if and only if $f - g \in A^+ P_k$. If $f \equiv 0$ then f is called hit.

Definition 2.3. A monomial x is said to be inadmissible if there exists the monomials

$y_1, y_2 ..., y_t$ such that $y_i < x,\ i = 1, 2, ..., t$ and $x - \sum_{i=1}^{t} y_i \in A^{+}P_k$.

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all admissible monomials in P_k is a minimal set of A-generators of P_k.

For later use, we set

$$\begin{aligned} P_k^0 &= \left\{ x = x_1^{a_1} x_2^{a_2} ... x_k^{a_k} \in P_k : \prod_{i=1}^{k} a_i = 0 \right\} \\ P_k^{+} &= \left\{ x = x_1^{a_1} x_2^{a_2} ... x_k^{a_k} \in P_k : \prod_{i=1}^{k} a_i > 0 \right\} \end{aligned} \qquad (14)$$

It is easy to see that P_k^o and P_k^{+} are the A-submodules of P_k. Furthermore, we have the following.

Proposition 2.4. We have a direct summand decomposition of the F_2-vector spaces

$$F_2 \otimes_A P_k = \left(F_2 \otimes_A P_k^0\right) \oplus \left(F_2 \otimes_A P_k^{+}\right) \qquad (15)$$

For a polynomial f in P_k, we denote by $[f]$ the class in $F_2 \otimes_A P_k$ represented by f.

3. PROOF OF MAIN THEOREM

From now on, we will denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k.

We first recall a result in [6] on the dimension of the vector space $(QP_5)_{39}$ as follows:

Since the squaring operation

$$\left(Sq_*^0\right)_{(k,m)} :\ \left(QP_k\right)_{2m+k} \to \left(QP_k\right)_m \qquad (16)$$

is an epimorphism of GL_k-modules, and using Proposition 2.4, we get

$$\begin{aligned} (QP_5)_{39} &\cong Ker\left(Sq_*^0\right)_{(5;17)} \oplus \operatorname{Im}\left(Sq_*^0\right)_{(5;17)} \cong \\ &\left(QP_5^0\right)_{39} \oplus \left(KerSq_{*\,(5;17)}^0 \cap \left(QP_5^{+}\right)_{39}\right) \oplus \left(QP_5\right)_{17} \end{aligned} \qquad (17)$$

For $1 \le i \le k$, define the homomorphism $f_i : P_{k-1} \to P_k$ of algebras by substituting

$$f_i\left(x_j\right) = \begin{cases} x_j, & \text{if } 1 \le \text{j<i} \\ x_{j+1}, & \text{if } \text{i} \le \text{j<k} \end{cases} \qquad (18)$$

Then, it can be easily seen that if B is a minimal set of generators for A-module P_{k-1} in degree n, then $f(B) = \bigcup_{i=1}^{k} f_i(B)$ is a minimal set of generators for A-module P_k^0 in degree n.

Based on the results in [13], we see that $\left|B_4(39)\right| = 225$ and therefore

$$\dim\left(QP_5^0\right)_{39} = \left|\bigcup_{i=1}^{5} f_i\left(B_4(39)\right)\right| = 915. \qquad (19)$$

$$\left|KerSq_{*\,(5;17)}^0 \cap \left(QP_5^{+}\right)_{39}\right| = 649 \qquad (20)$$

and $\left|B_5(17)\right| = 566$. From the above results, one gets $\left|B_5(39)\right| = 2130$ (see Phuc [8]).

Let $n \in \mathbb{N}$, we recall that the function $\mu : \mathbb{N} \to \mathbb{N}$ is given by: $\mu(0) = 0$ and

$$\begin{aligned} \mu(n) &= \min\left\{ m \in \mathbb{N} :\ n = \sum_{i=1}^{m}\left(2^{d_i} - 1\right),\ d_i > 0 \right\} \\ &= \min\left\{ m \in \mathbb{N} :\ \alpha(n+m) \le m \right\} \end{aligned} \qquad (21)$$

Here $\alpha(n)$ denotes the number of ones in dyadic expansion of n.

By direct computation, we easy to check that $\mu(39) = 3$, and $\alpha(39 + \mu(39)) = \mu(39)$. Moreover, we have

$$n = 11.2^{r+2} - 5 = 5\left(2^r - 1\right) + 39.2^r \qquad (22)$$

Hence, using Theorem 2, one get

$$\left|B_6\left(11.2^{r+2} - 5\right)\right| = \left(2^6 - 1\right)\left|B_5(39)\right| \qquad (23)$$

for any $r > 4$.

And therefore, there exists exactly 134190 admissible monomials of degree

$11.2^{r+2}-5$ in P_6, for any integer $r>4$. Main Theorem is proved.

REMARK

One of the major applications of hit problem is in surveying a homomorphism introduced by Singer in 1989. Singer [3] defined the algebraic transfer, which is a homomorphism

$$\varphi_k : Tor^A_{k;k+n}\left(F_2,F_2\right)\to\left(F_2\otimes_A P_k\right)^{GL_k}_n \qquad (24)$$

from the homology of the Steenrod algebra to the the subspace of $F_2\otimes_A P_k$ consisting of all the GL_k-invariant classes of degree n. It is a useful tool in describing the homology groups of the Steenrod algebra, $Tor^A_{k;k+n}\left(F_2,F_2\right)$.

Singer has indicated the importance of the algebraic transfer by showing that φ_k is an isomorphism with $k=1;2$ and at some other degrees with $k=3;\,4$ but he also disproved this for φ_5 at degree 9 and then gave the following conjecture.

Conjecture 4 (Singer [3]). The algebraic transfer φ_k is an epimorphism for any $k\geq 0$.

It could be seen from the work of Singer the meaning and necessity of the hit problem. In 1991, Boardman confirmed this again by using the modular representation theory of linear groups to show that φ_3 is an isomorphism. Recently, Hung and his collaborators have completely determined the image Tr_4, here Tr_4 is dual to φ_4. Furthermore, Hung proved in [7] that for any $k\geq 4$, φ_k is not an isomorphism in infinitely many degrees. However, it has not been known whether the algebraic transfer fails to be a monomorphism or fails to be an epimorphism.

Therefore, Singer's conjecture is still open for $k\geq 4$. Then, the results of hit problem are used to verify Singer's conjecture for the algebraic transfer.

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