

FAST UNCERTAINTY COMPUTATION FOR MULTI-ORDER SUBSPACE-BASED METHOD WITH QR DECOMPOSITIONS

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ABSTRACT

In Operational Modal Analysis, the Stochastic Subspace Identification (SSI) does not yield the exact system matrices, hence there is uncertainty in the estimates of modal parameters (natural frequencies, damping ratios, mode shapes). From that, it is necessary to evaluate the uncertainty of modal parameters. Recently, a new algorithm (multi-order subspace-based method) for determining modal parameters obtained from SSI has been proposed using QR decomposition (QRD). This new multi-order SSI method will partition the up-shifting observability matrix into smaller matrices, then the new state transition matrix is identified by using these smaller matrices. The article will propose a new algorithm for computing the uncertainty on modal parameters obtained from this multi-order SSI (MOSSI) based on QR decomposition (QRD). The new uncertainty algorithm only computes uncertainty bounds at the highest model order and gives bounds at lower orders by mean of selection matrices.

Keywords: *Stochastic Subspace Identification; Multi-order; QR Decomposition; Uncertainty Computation; Operational Modal Analysis.*

1. INTRODUCTION

The design and maintenance of mechanical structures subjected to noise and vibrations are important topics in mechanical engineering. It is an important component of comforts (cars and buildings) and contributes significantly to the safety related aspects of design and maintenance (aircrafts, aerospace vehicles and payloads, civil structures). Requirements from these application areas are numerous and demanding. Laboratory and in-operation tests are performed on the prototype structure, in order to get so-called modal models, i.e., to extract the modes and damping factors (these correspond to system poles), the mode shapes (corresponding eigenvectors), and loads. These results are used for updating the design model for a better fit to data, and sometimes for certification purposes (e.g., in flight domain opening for new aircrafts). Subspace-based linear system identification methods have been proven efficient for the system identification of mechanical systems, fitting a

linear model to (input/output or output-only) measurements taken from a system.

The estimation of modal parameters of structures can easily be carried out by using Stochastic Subspace Identification (SSI) methods on sensor measurements. From the early 80's, the family of subspace algorithms has been extensively studied (see in (Bauer, 2000) [1], (Bauer, 2002) [2], (Larimore, 1983) [3], (Van Overschee, 1996) [4], (Deistler, 1995) [5], (Chiuso, 2004) [6]) and has expanded rapidly. The family of subspace algorithms is growing in size and popularity (see in (Van Overschee, 1994) [7], (Verhaegen, 1994) [8], (Viberg, 1995) [9], (McKelvey, 1996) [10]).

Because of noise and model reduction, usual tools from linear system identification, such as the System Identification Toolbox by Matlab, are not used as such. In particular, recommended techniques from statistics to estimate the best model order (AIC, BIC, MDL, . . .) do not work at all. In order to retrieve the wanted large number of modes, an

even larger model order must be assumed while performing identification. This causes a number of spurious modes to appear in the identified models. Getting rid of these is the main issue in this context. Basically, all methods in use estimate a number of models of different orders and build a final model by fusing them in some way or another. So-called stabilization diagrams are a GUI-assisted way to support the engineer while performing this.

In (Dohler, 2011) [11], a multi-order system identification efficient algorithm to estimate the system matrices at multiple model orders has been derived, reducing the computational burden significantly, based on a pertinent use of appropriate QR decomposition (QRD).

The uncertainty on modal parameters appears for many reasons, e.g. finite number of data samples, undefined measurement noises, nonstationary excitations, nonlinear structure, model order reduction,... (,etc.) Then the system identification algorithms do not yield the exact system matrices. Practically, the statistical uncertainty of the obtained modal parameters at a chosen system order can be computed from the uncertainty of the system matrices, which depends on the covariance of the corresponding subspace matrix. Not knowing the model order yields to use empirical multiorder procedure such as the stabilization diagram (Peeters, 1999) [12], where modes of the system are assumed to stabilize when the model order increases.

In (Reynders, 2008) [13], (Lam, 2011) [14], it has been shown how uncertainty bounds for modal parameters can be determined from the covariances of the system matrices and the covariances of Hankel matrices. Many extensions are possible to this algorithm depending on the identification procedure of interest. Recently, uncertainty quantification has also been derived for Eigensystem-Realization-Algorithm, a class of subspace methods, see in (Lam, 2011) [14].

The current paper will expand on this and focus on an efficient multi-order uncertainty

computation for the QR based fast multi-order subspace algorithm (MOSA) (Dohler, 2011) [11]. Derivation of uncertainties is related to the work of (Pintelon, 2007) [15] and (Chang, 2001) [16], where perturbations of the QRD were studied.

2. STOCHASTIC SUBSPACE IDENTIFICATION (SSI)

This section describes the conventional SSI algorithm. The discrete time model in state-space form is:

$$\begin{cases} X_{k+1} = AX_k + V_{k+1} \\ Y_k = CX_k \end{cases} \quad (1)$$

with the state $X \in \mathbf{R}^n$, the output $Y \in \mathbf{R}^r$, the state transition matrix $A \in \mathbf{R}^{n \times n}$ and the observation matrix $C \in \mathbf{R}^{r \times n}$. The state noise V is unmeasured and assumed to be Gaussian, zero-mean, white.

Let r be the number of sensors, p and q be chosen parameters with $(p+1)r \geq qr \geq n$. From the output data, a matrix $H_{p+1,q} \in \mathbf{R}^{(p+1)r \times qr}$ is built according to a chosen SSI algorithm, see e.g. (Benveniste, 2007) [17] for an overview. The matrix $H_{p+1,q}$ will be called ‘‘Hankel matrix’’ in the following, and the SSI algorithm is chosen such that the corresponding Hankel matrix enjoys (asymptotically for a large number of samples) the factorization property

$$H_{p+1,q} = OZ \quad (2)$$

into the matrix of observability

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^p \end{bmatrix} \quad (3)$$

and a matrix Z depending on the selected SSI algorithm.

Let N be the number of available samples and $Y_k \in \mathbf{R}^r, \{k \in 1, \dots, N\}$ be the vector containing the sensor data. Then the ‘‘forward’’ and ‘‘backward’’ data matrices

$$Y_{p+1}^+ = \frac{1}{\sqrt{N_{pq}}} \begin{bmatrix} Y_{q+1} & Y_{q+2} & \cdots & Y_{N-p} \\ Y_{q+2} & Y_{q+3} & \cdots & Y_{N-p+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{q+p+1} & Y_{q+p+2} & \cdots & Y_N \end{bmatrix},$$

$$Y_q^- = \frac{1}{\sqrt{N_{pq}}} \begin{bmatrix} Y_q & Y_{q+1} & \cdots & Y_{N-p-1} \\ Y_{q-1} & Y_q & \cdots & Y_{N-p-2} \\ \vdots & \vdots & \ddots & \vdots \\ Y_1 & Y_2 & \cdots & Y_{N-p-q} \end{bmatrix} \quad (4)$$

are built ($N_{pq} = N - p - q$). For covariance-driven SSI, the Hankel matrix $H_{p+1,q}^{(\text{cov})} = Y_{p+1}^+ Y_q^{-T}$ is built, which enjoys the factorization property (2), where Z is the controllability matrix.

For simplicity, let p and q be given, skip the subscripts of $H_{p+1,q}$. The eigenstructure of the system is retrieved from a given matrix H .

The observability matrix O is obtained from a thin Singular Value Decomposition (SVD) of the matrix H and its truncation at the desired model order n :

$$H = U \Sigma V^T = [U_1 \ U_0] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_0^T \end{bmatrix} \quad (5)$$

$$O = U_1 \Sigma_1^{1/2} \quad (6)$$

Note that the singular values in $\Sigma_1 \in \mathbf{R}^{n \times n}$ must be non-zero and hence O is of full column rank. The observation matrix C is then found in the first block-row of the observability matrix O . T is transpose. The state transition matrix A is obtained from the shifting invariance property of O , namely as the least squares solution of

$$O^\uparrow A = O^\downarrow \quad (7)$$

$$\text{where } O^\uparrow = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}, \quad O^\downarrow = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^p \end{bmatrix}.$$

The O^\uparrow , O^\downarrow can be obtained from O by using selected matrices

$$O^\uparrow = S_1 O \quad (8)$$

$$O^\downarrow = S_2 O \quad (9)$$

with the selected matrices

$$S_1 = \begin{bmatrix} 0_{pr \times r} & I_{pr} \end{bmatrix} \quad (10)$$

$$S_2 = \begin{bmatrix} I_{pr} & 0_{pr \times r} \end{bmatrix} \quad (11)$$

The eigenstructure (λ, ϕ_λ) results from

$$\det(A - \lambda I) = 0, A \phi_\lambda = \lambda \phi_\lambda, \phi_\lambda = C \phi_\lambda \quad (12)$$

where λ ranges over the set of eigenvalues of A . From λ , the natural frequency and damping ratio are obtained, and ϕ_λ is corresponding mode shape.

Notice that the least squares solution of (7) can be expressed as

$$A = R^{-1} Q^T O^\downarrow \quad (13)$$

where the thin QRD comes from

$$O^\uparrow = QR \quad (14)$$

$R \in \mathbf{R}^{n \times n}$ being invertible, and $Q \in \mathbf{R}^{pr \times n}$ being orthogonal, this formulation is more stable. Moreover, it will be used for the fast MOSA.

3. DESCRIPTIONS OF SSI UNCERTAINTY QUANTIFICATION ALGORITHM

The uncertainty ΔA and ΔC of system matrices A and C (see (Reynders, 2008) [13]) are connected to the uncertainty of Hankel matrix through a Jacobian matrix $J_{A,C}$

$$\begin{bmatrix} \text{vec} \Delta A \\ \text{vec} \Delta C \end{bmatrix} = \begin{bmatrix} J_A \\ J_C \end{bmatrix} \text{vec} \Delta H = J_{A,C} \text{vec} \Delta H \quad (15)$$

The uncertainty of modal parameters (natural frequency f , damping ratio d , mode shape ϕ) is derived from

$$\Delta f_\mu = J_{f_\mu} \begin{bmatrix} \text{vec} \Delta A \\ \text{vec} \Delta C \end{bmatrix}; \Delta d_\mu = J_{d_\mu} \begin{bmatrix} \text{vec} \Delta A \\ \text{vec} \Delta C \end{bmatrix}; \quad (16)$$

$$\Delta \phi_\mu = J_{\phi_\mu} \begin{bmatrix} \text{vec} \Delta A \\ \text{vec} \Delta C \end{bmatrix}$$

The Jacobian matrices J_{f_μ} , J_{d_μ} , J_{ϕ_μ} are computed for each mode μ . Finally, the covariances of the modal parameters are obtained as

$$\text{cov}(f_\mu) = J_{f_\mu} J_{A,C} \text{cov}(\text{vec} H) J_{A,C}^T J_{f_\mu}^T$$

$$\text{cov}(d_\mu) = J_{d_\mu} J_{A,C} \text{cov}(\text{vec} H) J_{A,C}^T J_{d_\mu}^T \quad (17)$$

$$\text{cov}(\phi_\mu) = J_{\phi_\mu} J_{A,C} \text{cov}(\text{vec} H) J_{A,C}^T J_{\phi_\mu}^T$$

where $\text{cov}(\text{vec} H)$ is the covariance of the vectorized Hankel matrix. After retrieving the uncertainties on the system matrices A and C , the calculation of the uncertainties on the frequency and damping is straightforward. However, for the mode shape, there is an issue of normalization as each one is defined up to an unknown constant. This was addressed in (Dohler, 2010) [18].

4. DIFFERENTIATING THE QR-BASED LEAST SQUARES EQUATION

The objective of this section is to investigate the uncertainty related to the least squares problem

$$O^\uparrow A = O^\downarrow \quad (18)$$

Where the following QRD is introduced

$$O^\uparrow = QR \quad (19)$$

$$A = R^{-1} Q^T O^\downarrow \quad (20)$$

The sensitivity of system matrix A expresses as

$$\Delta A = -R^{-1} (\Delta R) R^{-1} Q^T O^\downarrow + R^{-1} (\Delta(Q^T)) O^\downarrow + R^{-1} Q^T \Delta O^\downarrow \quad (21)$$

This can also be written in vectorized form as

$$\text{vec} \Delta A = - \left[\left((O^\downarrow)^T Q R^{-1T} \right) \otimes R^{-1} \right] \text{vec}(\Delta R)$$

$$+ \left[(O^\downarrow)^T \otimes R^{-1} \right] \text{vec}(\Delta Q^T)$$

$$+ \left[I_n \otimes (R^{-1} Q^T) \right] \text{vec}(\Delta O^\downarrow)$$

$$= - \left[\left((O^\downarrow)^T Q R^{-1T} \right) \otimes R^{-1} \right] \text{vec}(\Delta R)$$

$$+ \left[(O^\downarrow)^T \otimes R^{-1} \right] P_Q \text{vec}(\Delta Q)$$

$$+ \left[I_n \otimes (R^{-1} Q^T) \right] \text{vec}(\Delta O^\downarrow)$$

$$= - \left[\left((O^\downarrow)^T Q R^{-1T} \right) \otimes R^{-1} \right] \text{vec}(\Delta R)$$

$$+ \left[(O^\downarrow)^T \otimes R^{-1} \right] P_Q \text{vec}(\Delta Q)$$

$$+ \left[I_n \otimes (R^{-1} Q^T) \right] (I_n \otimes S_2) \text{vec}(\Delta O) \quad (22)$$

where P_Q is a matrix that can permute $\text{vec}(\Delta Q)$ to $\text{vec}(\Delta(Q^T))$.

5. UNCERTAINTY QUANTIFICATION FOR THE R MATRIX

Following the lines of (Chang, 1997) [19], the sensitivity of matrix R is derived as

$$\Delta R = (\Delta R_{up}) R \quad (23)$$

where

$$\Delta R_{up} = up \left[Q^T (\Delta O^\uparrow) R^{-1} + R^{-1T} (\Delta(O^\uparrow)^T) Q \right] \quad (24)$$

The up operator is defined as

$$up(X) = \begin{bmatrix} x_{11}/2 & x_{12} & \cdots & x_{1n} \\ 0 & x_{22}/2 & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{nn}/2 \end{bmatrix} \quad (25)$$

For any square matrix $C = (c_{ij}) = (c_1, \dots, c_n)$, denote by $c_j^{(i)}$ the vector of the first i elements of c_j . With

this, we define $uvec(C) = \begin{bmatrix} c_1^{(1)} \\ c_2^{(2)} \\ \vdots \\ c_n^{(n)} \end{bmatrix}$. It is the

vector formed by stacking the columns of the upper triangular part of C into one long vector.

Remark:

The upper-vectorization of the matrix of derivative of R can be determined by

$$\begin{aligned} uvec(\Delta R) &= W_R^{-1} Z_R (I_n \otimes Q^T) uvec \Delta O^\uparrow \\ &= W_R^{-1} Z_R (I_n \otimes Q^T) (I_n \otimes S_1) uvec \Delta O \end{aligned} \quad (26)$$

and the vectorized perturbation of R is selected from the upper-vectorization of R derivative as

$$uvec \Delta R = S_R uvec \Delta R \quad (27)$$

where matrices W_R and Z_R are defined in (Chang, 1997) [19].

6. UNCERTAINTY QUANTIFICATION FOR THE Q MATRIX

The perturbation of matrix Q is written as

$$\begin{aligned} \Delta Q &= (\Delta O^\uparrow) R^{-1} - Q (\Delta R_{up}) \\ &= (\Delta O^\uparrow) R^{-1} - Q (\Delta R_{up}) R R^{-1} \\ &= (\Delta O^\uparrow) R^{-1} - Q (\Delta R) R^{-1} \end{aligned} \quad (28)$$

Vectorizing ΔQ leads to

$$\begin{aligned} uvec \Delta Q &= (R^{-1T} \otimes I_{pr}) (I_n \otimes S_1) uvec \Delta O \\ &\quad - (R^{-1T} \otimes Q) uvec \Delta R \end{aligned} \quad (29)$$

7. UNCERTAINTY QUANTIFICATION RELATIVELY TO H

Deriving the sensitivity of C in vectorized form leads to

$$uvec \Delta C = (I_n \otimes S_C) uvec \Delta O \quad (30)$$

Deriving the sensitivity of O^\downarrow in vectorized form leads to

$$uvec \Delta O^\downarrow = (I_n \otimes S_2) uvec \Delta O \quad (31)$$

Finally, the perturbation of observability can be connected to the uncertainty of Hankel matrix as

$$uvec \Delta O = (B + C) uvec \Delta H \quad (32)$$

where

$$B = \left(I_n \otimes \left(\frac{1}{2} U_1 \Sigma_1^{-1/2} \right) \right) S_3 \begin{bmatrix} (v_1 \otimes u_1)^T \\ \vdots \\ (v_n \otimes u_n)^T \end{bmatrix} \quad (33)$$

$$C = \left(\Sigma_1^{1/2} \otimes I_{(p+1)r} \right) L_1 \begin{bmatrix} B_1^\dagger C_1 \\ \vdots \\ B_n^\dagger C_n \end{bmatrix} \quad (34)$$

$$S_3 = \sum_{s=1}^n E_{(s-1)n+s,n}^{n^2 \times n} \quad (35)$$

$$L_1 = I_n \otimes \begin{bmatrix} I_{(p+1)r} & 0_{(p+1)r \times qr} \end{bmatrix} \quad (36)$$

$$B_j = \begin{bmatrix} I_{(p+1)r} & -\frac{H}{\sigma_j} \\ -\frac{H^T}{\sigma_j} & I_{qr} \end{bmatrix} \quad (37)$$

$$C_j = \frac{1}{\sigma_j} \begin{bmatrix} v_j^T \otimes (I_{(p+1)r} - u_j u_j^T) \\ (u_j^T \otimes (I_{qr} - v_j v_j^T)) P \end{bmatrix} \quad (38)$$

$$P = \sum_{k_1=1}^{(p+1)r} \sum_{k_2=1}^{qr} E_{k_1 k_2}^{(p+1)r \times qr} \otimes E_{k_2 k_1}^{qr \times (p+1)r} \quad (39)$$

σ_j is eigenvalue at system order j $\{j \in 1, \dots, n\}$. u_j (resp. v_j) is column number j of U_1 (resp. V_1). $E_{k_1 k_2}^{(p+1)r \times qr}$ is a $(p+1)r \times qr$ matrix whose element is 1 at position (k_1, k_2) and zero elsewhere.

8. MULTI-ORDER SSI (MOSSI)

In many practical applications, the true system order n is unknown and it is common to do the system identification for models at different system orders $n = n_j, j = 1, \dots, t$, with

$$1 \leq n_1 < n_2 < \dots < n_t \leq \min\{pr, qr_0\} \quad (40)$$

And where t is the number of models to be estimated. The choice of the model orders $n_j, j = 1, \dots, t$, is up to the user and also depends on the problem. For example, $n_j = j + c$ or $n_j = 2j + c$ with some constant c can be chosen.

The following notation for specifying these different system orders is introduced and used throughout this paper. Let $O_j \in \mathbf{R}^{(p+1)r \times n_j}, A_j \in \mathbf{R}^{n_j \times n_j}$ and $C_j \in \mathbf{R}^{r \times n_j}$ be the observability, state transition, and observation matrix at model order $n_j, j \in \{1, \dots, t\}$, respectively. Let furthermore O_j^\uparrow and O_j^\downarrow be the first respective last p block rows of O_j .

9. COMPUTATION OF THE SYSTEM MATRICES

The system matrix A_j is the solution of the least squares problem at a chosen model order n_j . A common numerically stable way to solve it is

$$A_j = O_j^{\uparrow\uparrow} O_j^\downarrow \quad (41)$$

where \dagger denotes the Moore-Penrose pseudo-inverse.

A more efficient and also numerically stable way to solve it (see also (Golub, 1996) [20]) is to do the thin QRD

$$O_j^\dagger = Q_j R_j \quad (42)$$

where $Q_j \in \mathbf{R}^{pr \times n_j}$ is a matrix with orthogonal columns and $R_j \in \mathbf{R}^{n_j \times n_j}$ is

upper triangular. R_j is assumed to be of full rank, which is reasonable as O_j is of full column rank.

With

$$S_j = Q_j^T O_j^\downarrow \quad (43)$$

$S_j \in \mathbf{R}^{n_j \times n_j}$, the solution of the least squares solution is

$$A_j = R_j^{-1} S_j \quad (44)$$

The observation matrix C_j is found in the first block row of O_j .

Theorem 1 (see (Dohler, 2011) [11]):

Let O_t, Q_t, R_t , and S_t be given at maximal desired model order n_t with

$$O_t^\uparrow = Q_t R_t, S_t = Q_t^T O_t^\downarrow, A_t = R_t^{-1} S_t \quad (45)$$

Such that A_t is the least squares solution

$$O_t^\uparrow A_t = O_t^\downarrow \quad (46)$$

Let $j \in \{1, \dots, t-1\}$, and let R_t^{-1} and S_t be partitioned into blocks

$$R_t^{-1} = \begin{bmatrix} R_j^{(11)} & R_j^{(12)} \\ 0 & R_j^{(22)} \end{bmatrix}, S_t = \begin{bmatrix} S_j^{(11)} & S_j^{(12)} \\ S_j^{(21)} & S_j^{(22)} \end{bmatrix} \quad (47)$$

where $R_j^{(11)}, S_j^{(11)} \in \mathbf{R}^{n_j \times n_j}$. Then, the state transition matrix A_j at model order n_j , which is the least squares solution of

$$O_j^\uparrow A_j = O_j^\downarrow \quad (48)$$

satisfies

$$A_j = R_j^{(11)} S_j^{(11)} \quad (49)$$

10. SYSTEM MATRIX UNCERTAINTY AT ORDER T

The perturbation of S_t is expanded to

$$\begin{aligned}\Delta S_i &= \Delta(Q_i^T O_i^\downarrow) \\ &= (\Delta Q_i^T) O_i^\downarrow + Q_i^T \Delta O_i^\downarrow\end{aligned}\quad (50)$$

Vectorizing ΔS_i leads to

$$\begin{aligned}\text{vec}\Delta S_i &= \left((O_i^\downarrow)^T \otimes I_n \right) \text{vec}(\Delta Q_i^T) \\ &+ (I_n \otimes Q_i^T) \text{vec}\Delta O_i^\downarrow \\ &= \left((O_i^\downarrow)^T \otimes I_n \right) P_{Q_i} \text{vec}(\Delta Q_i) \\ &+ (I_n \otimes Q_i^T) (I_n \otimes S_2) \text{vec}\Delta O_i\end{aligned}\quad (51)$$

11. SYSTEM MATRIX UNCERTAINTY AT ORDER J

At a given order j , let $R = R_j^{(11)}$ and $S = S_j^{(11)}$. The state matrix $A = A_j$ is described by the QRD

$$A_j = R_j^{(11)} S_j^{(11)} = (S_R^{(1)} R_i^{-1} S_R^{(2)}) (S_S^{(1)} S_i S_S^{(2)}) \quad (52)$$

where $S_R^{(1)}$, $S_R^{(2)}$, $S_S^{(1)}$ and $S_S^{(2)}$ are selection matrices depending of order j .

The perturbation of state matrix can be rewritten for the MOSSI using some selection matrices

$$\begin{aligned}\Delta A_j &= S_R^{(1)} \left(\Delta(R_i^{-1}) \right) S_R^{(2)} S_S^{(1)} S_i S_S^{(2)} \\ &+ S_R^{(1)} R_i^{-1} S_R^{(2)} S_S^{(1)} (\Delta S_i) S_S^{(2)} \\ &= -S_R^{(1)} R_i^{-1} (\Delta R_i) R_i^{-1} S_R^{(2)} S_S^{(1)} S_i S_S^{(2)} \\ &+ S_R^{(1)} R_i^{-1} S_R^{(2)} S_S^{(1)} (\Delta S_i) S_S^{(2)}\end{aligned}\quad (53)$$

As previously, the vectorization expression of ΔA is expressed by mean of Kronecker products as

$$\begin{aligned}\text{vec}\Delta A_j &= \left((S_S^{(2)T} S_i^T S_S^{(1)T} S_R^{(2)T} R_i^{-1T}) \otimes (S_R^{(1)} R_i^{-1}) \right) \text{vec}\Delta R_i \\ &+ (S_S^{(2)T} \otimes (S_R^{(1)} R_i^{-1} S_R^{(2)} S_S^{(1)})) \text{vec}\Delta S_i\end{aligned}\quad (54)$$

The vectorized perturbation of C can be described as

$$\text{vec}\Delta C_j = (S_{C_j}^T \otimes I_r) (I_n \otimes S_C) \text{vec}\Delta O \quad (55)$$

with

$$S_{C_j} = \begin{bmatrix} I_{n_j} \\ \mathbf{0}_{(n-n_j) \times n_j} \end{bmatrix} \quad (56)$$

$$S_C = \begin{bmatrix} I_r & \mathbf{0}_{r \times pr} \end{bmatrix} \quad (57)$$

Finally, the uncertainty of system matrices can be joined together

$$\begin{bmatrix} \text{vec}\Delta A_j \\ \text{vec}\Delta C_j \end{bmatrix} = \begin{bmatrix} J_A \\ J_C \end{bmatrix} \text{vec}\Delta H = J_{A,C} \text{vec}\Delta H \quad (58)$$

Then, the covariances of the modal parameters are obtained as

$$\begin{aligned}\text{cov}(f_\mu) &= J_{f_\mu} J_{A,C} \text{cov}(\text{vec}H) J_{A,C}^T J_{f_\mu}^T \\ \text{cov}(d_\mu) &= J_{d_\mu} J_{A,C} \text{cov}(\text{vec}H) J_{A,C}^T J_{d_\mu}^T \\ \text{cov}(\phi_\mu) &= J_{\phi_\mu} J_{A,C} \text{cov}(\text{vec}H) J_{A,C}^T J_{\phi_\mu}^T\end{aligned}\quad (59)$$

12. NUMERICAL EXAMPLES

The S101 bridge (Siringoringo, 2010) [21] connected Salzburg – Vienna carriage way in Austria. It is a post-tensioned concrete bridge with a main span of 32 m, side spans of 12 m, and the width of 6.6 m. The deck is continuous over the piers. This bridge, constructed in 1960, has been a typical overpass bridge in Austria national highway. In the current paper, the ambient vibration data is collected on 15 sensors. The original sampling frequency is 500 Hz with 165000 time samples available. The data is decimated to 35.7 Hz and only five modes are taken into account.



Figure 1. S101 bridge

The summary of the frequencies and damping ratios of the five identified modes is given in Table 1 and Table 2, for both conventional SSI (Peeters, 1999) [12] and QR-based SSI (QRSSI) (Dohler, 2011) [11]. The obtained frequencies (as well as damping ratios) of conventional SSI and QRSSI are almost similar. For the computation of uncertainty bounds on modal parameters, 18 time lags, leading to $p + 1 = q = 9$, and 40 model orders are utilized.

Table 1. Frequency uncertainty quantification with SSI and QRSSI

Mode	Frequency f (Hz)	Uncertainty bounds (%)	
		SSI	QRSSI
1	4.046	0.108	0.108
2	6.279	0.112	0.111
3	9.695	0.564	0.513
4	13.276	0.230	0.209
5	15.659	0.521	0.526

Table 2. Damping-ratio uncertainty quantification with SSI and QRSSI

Mode	Damping ratio d (%)	Uncertainty bounds (%)	
		SSI	QRSSI
1	0.835	23.290	22.488
2	0.633	17.542	17.520
3	1.218	18.402	17.555
4	1.542	44.652	42.156
5	1.645	36.871	37.075

In Table 1 and Table 2, the uncertainty bounds for the natural frequencies and damping ratios of the five modes are presented, respectively. Computations of uncertainty bounds are relatively similar for both algorithms. Frequency uncertainties are much smaller than those of damping ratios. This is a priori coherent with statistical theory, since the lower bound of the covariance given by Fisher information matrix is smaller for the frequencies than for the damping ratios (Gersch, 1974) [22].

13. CONCLUSIONS

In this paper, the uncertainty quantification for modal parameters is derived and implemented for both the conventional SSI (Peeters, 1999) [12] and the new QRSSI derived in (Dohler, 2011) [11]. Both algorithms give comparable results on the ambient vibration data of the S101 overpass bridge. It is observed that the uncertainty for QRSSI is relatively similar to the uncertainty associated with the former. The uncertainty computation for the QRSSI inherits the quality of the corresponding identification algorithm derived in (Dohler, 2011) [11], i.e. it only computes uncertainty bounds at the highest model order and gives bounds at lower orders by means of selection matrices.

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