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Introduction to Algebraic Deformation Theory and The Case of k-Linear Categories

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ABSTRACT

Deformation theory is a branch of mathematics which studies how mathematical objects, such as algebraic varieties, schemes, algebras, or categories, can be deformed "continuously" depending on a space of parameter while preserving certain algebraic or geometric structures. Algebraic deformation theory, which was pioneered by Murray Gerstenhaber in 1960s-1970s, established its role as a cornerstone in modern mathematics and theoretical physics. This theory provides a powerful framework for understanding the subtle variations and deformations of mathematical and physical objects depending on a parameter space. Lying at the interface of algebra, geometry and topology this theory has been being studied extensively worldwide and obtained many applications in various areas of mathematics and theoretical physics, such as the study of Calabi-Yau manifolds, mirror symmetry, quantum physics. In such context, this article aims to introduce this important mathematical theory to the community of Vietnamese mathematicians in a hope to bring more attention of Vietnamese mathematicians and math students to this vibrant research area. We also expect that this topic will be taught at universities in Vietnam in the near future.

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1. Introduction

In mathematics, deformation theory studies how a mathematical objects (like varieties, manifolds, algebras, etc.) in certain space can be varied dependently on points of a parameter space. Foundations of deformations theory was initiated in the works of A. Grothendieck, K. Kodaira, D.C. Spencer in the 1950's. In later years, in the ground breaking paper [5], M. Gerstenhaber introduced algebraic deformation theory for algebras and rings. In the following years, Gerstenhaber's deformation theory was studied intensively by a large number of mathematicians in various areas of mathematics. This theory has obtained far-reaching applications across diverse areas of mathematics and theoretical physics. Nowadays, algebraic deformation theory stands as a cornerstone at the interface of several fundamental research areas in mathematics including algebra, geometry, topology and mathematical physics.

Beyond the study about how algebraic structures such as associative algebras, Lie algebras, and morphisms between them behave under continuous perturbations, in recent years, various flavours of deformation theory have been studied intensively including:

(1) Deformations of Poisson structures were studied by D. Kaledin, M. Kontsevich, D. Tamarkin, S. Merkulov. Most remarkable is the work on deformation quantization of Poisson structure [10], which brought Kontsevich to the Fields medal award in 1998.

(2) Deformations of abelian categories, presheaves, prestacks, were studied by M. Van den Bergh, W. Lowen, H. Dinh Van, L. Hermans, etc. in [12], [14], [13], [1], [2], [3]

(3) Deformations of monoidal categories were studied by D. N. Yetter, T. Shrestha [19].

(4) Deformations of operads were studied by M. Markl, S. Merkulov, B. Vallette, ect. in [18], [15], [17].

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In the context that the theory of deformations is being studied extensively world-wide, to the best of my knowledge, there are very few Vietnamese mathematicians doing research in this field. Therefore, the main purpose of this article is to introduce the basic knowledge of deformation theory to mathematicians in Vietnam and hopefully to attract more Vietnamese mathematicians to participate in this vibrant research area.

This paper is organized in three Sections. The main part is Section 2, in this section we present the basic knowledge of deformation theory, we introduce the Hochschild cohomology and clarify its indispensable role in classifying deformations. There are many references on this topic such as [7], [6],[4], [11], [16]. Among these references, perhaps the long paper by Gerstenhaber-Schack [7] is the best overview on the subject. However, we writing this note at a more elementary level in the hope that it will make the general theory more accessible to the non-expert. After basic theory of algebraic deformation, in Section 3, we show that how this theory is applied to the case of *k*-linear categories. In particular, we show that first order deformations of a *k*-linear category is controlled by its second Hochschild cohomology group.

2. Deformation for associative algebras

In this section, we present basic notions and properties of algebraic deformation pioneered by Gerstenhaber in 1960s. There are numerous references in scope and depth about this topic in the literature, such as [7], [6], [4], [11], [16]. Of those, the paper [7] may be the best overview on the subject. Thorough this paper, k is a field and A is an algebra over k.

2.1. Hochschild cohomology

Hochschild (co)homology is introduced in the seminal paper [8] by G. Hochschild. Later it become a fundamental tool in study deformation theory. For those results presented in this subsection which we do not prove, the reader can find their proofs in [8], [16].

Definition 2.1. Let M be an A-bimodule. The group of n-Hochschild cochains is

$$C^n(A,M) = hom_k(A^{\otimes n}, M) \tag{1}$$

and the differential $d^n : C^n(A,M) \to C^{n+1}(A,M)$ is defined as:

$$d^n(f)(a_0, a_1, \dots, a_n) = a_0 f(a_1, \dots, a_n) + \sum_{i=0}^{n-1} (-1)^{i+1} f(a_1, \dots, a_i a_{i+1}, \dots, a_n) \\ +(-1)^{n+1} f(a_0, \dots, a_{n-1}) a_n \tag{2}$$

where $f \in C^n(A,M)$ and $a_0, a_1, \dots, a_n$ in A. It is straight forward to check that $d^{n+1} \circ d^n = 0$ for any $n \geq 1$. The n-Hochschild cohomology group is

$$\mathrm{HH}^n(A, M) = \mathrm{H}^n(\mathrm{C}^\bullet(A, M)) = \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}) \tag{3}$$

Example 2.2. For a cochain $f \in \mathbf{C}^2(A) := \mathbf{C}^2(A, A)$, we have that $d^2(f)(a, b, c) = af(b, c) - f(ab, c) + f(a, bc) - f(a, b)c$.

Lemma 2.3. $\mathrm{HH}^0(A, M) \simeq \{m \in M : am = ma, \forall a \in A\}$. *In special case* $M = A$, *it follows that* $\mathrm{HH}^0(A)$ *is the center of* A.

Proof. We have $\mathbf{C}^0(A, M) = \mathrm{hom}_k(k, M) = M$ and $\mathbf{C}^1(A, M) = \mathrm{hom}_k(A, M)$ Take a cochain $m \in M$. $m \in \ker(d^0)$ if $d^0(m)(a) = am - ma = 0$ or $am = ma$. □

Lemma 2.4. $\mathrm{HH}^1(A, M) \simeq \mathrm{Der}(A, M)/\mathrm{InnDer}(A, M)$

Proof. Consider a cochain $f \in \mathbf{C}^1(A, M)$. Assume that $f \in \mathrm{Ker}d^1$, then $d^1(f)(a, b) = af(b) - f(ab) + f(a)b = 0, \forall a, b \in A$. It follows that $f(ab) = af(b) + f(a)b$, or $f \in \mathrm{Der}(A, M)$. The image of d^0 is given by function $g \in \mathrm{hom}_k(A, M)$ where $g(a) = am - ma$. It is easily to verify that $g(ab) = g(a)b + ag(b)$. These such function forms the set of inner derivative $\mathrm{InnDer}(A, M)$. So $\mathrm{Im}(d^0) = \mathrm{InnDer}(A, M)$. □

Definition 2.5. (Gerstenhaber bracket). Let $f \in \mathbf{C}^m(A, A)$ and $g \in \mathbf{C}^n(A, A)$. We define Gerstenhaber bracket $[f, g] \in \mathbf{C}^{m+n-1}$ as follows.

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f \tag{4}$$

where

$$(f \circ g)(a_1, \dots, a_{m+n-1}) = \sum_{i=1}^{m+n-1} (-1)^{(m-1)(i-1)} f(a_1, \dots, a_{i-1}, g(a_i, \dots, a_{i+n-1}), a_{i+n}, \dots, a_{m+n-1}) \tag{5}$$

for any $a_1, \dots, a_{m+n-1} \in A$.

Proposition 2.6. *Let* $f \in \mathbf{C}^m(A, A)$, $g \in \mathbf{C}^n(A, A)$ *and* $h \in \mathbf{C}^p(A, A)$. *Then*

(1) $[f, g] = -(-1)^{(m-1)(n-1)}[g, f]$.

(2) $(-1)^{(m-1)(p-1)}\big[f, [g, h]\big] + (-1)^{(n-1)(m-1)}\big[g, [h, f]\big] + (-1)^{(p-1)(n-1)}\big[h, [f, g]\big] = 0$.

(3) $d^{m+n-1}([f, g]) = [d^m(f), g] + (-1)^{m-1}[f, d^n(g)]$

Corollary 2.7. *Let* $\mathbf{C}^\bullet(A) = \oplus_{n=0}^{\infty}\mathbf{C}^n(A, A)$, *and let* $\delta(f) = (-1)^{m-1}d^m(f)$ *for* $f \in C^m(A)$. *Then* $(\mathbf{C}^\bullet(A), [-,-], \delta)$ *is a graded Lie algebra.*

2.2. Deformation theory

Deformation theory for algebras was developed by Gerstenhaber in 1960's, this theory is closely parallel to deformation theory for complex analytic structures initiated by Kodaira and Spencer [9].

Let A be an algebra over the field k. Denote by $A[[t]]$ the algebra of formal power series $\sum_{i=0}^{\infty} a_i t^i$ with coefficients $a_i \in A$. The addition and multiplication are defined as follows:

$$(\sum_{i=0}^{\infty} a_i t^i) + (\sum_{j=0}^{\infty} b_j t^j) = \sum_{i=0}^{\infty} (a_i + b_i) t^n \tag{6}$$

$$(\sum_{i=0}^{\infty} a_i t^i)(\sum_{j=0}^{\infty} b_j t^j) = \sum_{n\geq 0} (\sum_{i+j=n} a_i b_j) t^n \tag{7}$$

A new multiplication μ_t on $A[[t]]$ can be defined as:

$$\mu_t(a, b) = \sum_{i=0}^{\infty} \mu_i(a, b) t^i = ab + \mu_1(a, b)t + \mu_2(a, b) + \cdots \tag{8}$$

where $\mu_i \in \mathbf{C}^2(A, A)$, μ_0 is exactly the multiplication of A, and $a, b \in A$. This multiplication is defined for elements in A but it is extended linearly to all elements in $A[[t]]$.

Definition 2.8. We define deformations of A as follows:

• A formal deformation of A is the pair $(A[[t]], \mu_t)$ in which the multiplication μ_t is associative that make $A[[t]]$ into an algebra over $k[[t]]$.

• The deformation $(A[[t]], \mu_t)$ in which $\mu_i = 0$ for $i \geq 1$ ($\mu_t = \mu_0$) is called trivial deformation of A.

• If we set $t^2 = 0$ then $(A[[t]] = A \oplus At, \mu_t = \mu_0 + \mu_1 t)$ is called first order deformation of A.

Example 2.9. Let $A = k[t]$. Set $\mu_i(x^m, x^n) = \frac{m^i n^i}{i!} x^{m+n}$, we define $\mu_t = \sum_{i=0}^{\infty} \mu_i t^i$. By direct computations, one can show that μ_t is an associative multiplication of $A[[t]]$, thus $(A[[t]], \mu_t)$ is a deformation of the algebra A.

Let look at the associativity of μ_t:

$$\mu_t(\mu_t(a, b), c) = \mu_t(a, \mu_t(b, c)), \ \forall a, b, c \in A \tag{9}$$

Expand the left hand side we have that

$$\begin{aligned} \mu_t(\mu_t(a, b), c) &= \mu_t(\sum_{i=0}^{\infty} \mu_i(a, b)t^i, c) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \mu_j(\mu_i(a, b), c) t^{i+j} \\ &= \sum_{n=0}^{\infty} (\sum_{i+j=n} \mu_j(\mu_i(a, b), c) t^n \end{aligned} \tag{10}$$

Similarly expansion for the right hand side

$$\mu_t(a,\mu_t(b,c)) = \mu_t(\sum_{i=0}^{\infty} a,\mu_i(b,c)t^i) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\mu_j(a,\mu_i(b,c))t^{i+j} = \sum_{n=0}^{\infty}(\sum_{i+j=n}^{\infty}\mu_j(a,\mu_i(b,c))t^n \quad (11)$$

Let μ_n be the first nonzero coefficient after μ_0 in the expansion $\mu_t \sum_{i=0}^{\infty}\mu_i t^i$, then μ_n is called the infinitesimal of μ. Equating the coefficients of t^n of both sides of associativity equation (9) we obtain that:

$$a\mu_n(b, c) - \mu_n(ab, c) + \mu_n(a, bc) - \mu_n(a, b)c = 0 \quad (12)$$

for all $a, b, c \in A$. The connection to cohomology can be seen here $d^2(\mu_n) = 0$, this implies that μ_n is a Hochschild 2-cocycles. This leads us to the first important result in deformation theory

Theorem 2.10. *Let* $(A[[t]],\mu_t = \mu_0 + \sum_{i\geq n}^{\infty}\mu_i t^i)$ *be a deformation of the algebra A, then* μ_n *is a Hochschild 2-cocycle.*

Theorem 2.11. *There is one-one corresponding between first order deformation of an algebra A and its second Hochschild cohomology group* HH^2(*A*).

Proof. Let $A[t] = A \oplus At$, and let $\mu = \mu_0 + \mu_1 t, \mu' = \mu_{0+}\mu_1' t$ be two first order deformations of *A*. The equation (12) tell us that both μ_1 and μ_1' are two Hochschild 2-cochains. Now assume $\phi = \phi_0 + \phi_1 t: (A[t], \mu = \mu_0 + \mu_1 t) \to (A[t], \mu' = \mu_0 + \mu_1' t)$, is an isomorphism as in Definition 2.15, in which $\phi(a) = a + \phi_1(a)t$ for all $a \in A$. Condition $\mu'(\phi) = \phi(\mu)$ is expanded as:

$$(a + \phi_1(a)t)(b + \phi_1(b)t) + \mu_1'(a + \phi_1(a)t, b + \phi_1(b)t) = ab + \mu_1(a,b)t + \phi_1(ab)t \quad (12)$$

Equating the coefficients of *t* in both sides we get:

$$\mu_1(a,b) - \mu_1'(a,b) = a\phi_1(b) - \phi_1(ab) + \phi_1(a)b \quad (14)$$

This means that $\mu_1 - \mu_1' \in \mathrm{Im}(d^1)$. The proof is completed. □

Further more, inspecting coefficients of t^2 in both sides of equation (9) gives

$$\mu_1(\mu_1(a,b),c) - \mu_1(a,\mu_1(b,c)) = a\mu_2(b,c) - \mu_2(ab,c) + \mu_2(a,bc) - \mu_2(a,b)c \quad (15)$$

Equivalently,

$$\mu_1(\mu_1(a,b),c) - \mu_1(a,\mu_1(b,c)) = d^2(\mu_2) \quad (16)$$

The left hand side of this equation is called the 2nd obstruction. Working out further on the coefficient of t^n of equation (9) we obtain *n*-obstruction to lift to μ_n:

$$d^2(\mu_n)(a,b,c) = \sum_{i=1}^{n-1}\left(\mu_i(\mu_{n-i}(a,b),c) - \mu_i(a,\mu_{n-i}(b,c))\right) \quad (17)$$

Lemma 2.12. *Let* f, g *be Hochschild 2-cocyles in* **C**^{2}(*A*). *Define the Hochschild 3-cochain* $U(a,b,c) = f(g(a,b),c) - f(a,g(b,c))$ *for all a, b, c in A. Then*

$$d^3(U)(a,b,c,d) = g(a,b)f(c,d) - f(a,b)g(c,d) \quad (18)$$

where a, b, c, d $\in A$.

Proof. We have

$$d^3(U)(a,b,c,d) = aU(b,c,d) - U(ab,c,d) + U(a,bc,d) - U(a,b,cd) + U(a,c,b)d.$$

Firstly, let evaluate the right right hand side of this differential equation for $f(g(*,*)*)$:

$$L = af(g(b,c),d) - f(g(ab,c),d) + f(g(a,bc),d) - f(g(a,b),cd) + f(g(a,c),b)d \quad (19)$$

$$= f(ag(b,c) - g(ab,c) + g(a,bc),d) - f(g(a,b),cd) + f(g(a,c),b)d \quad (20)$$

Since g is a 2-cocycle, we have that

$$ag(b,c) - g(ab,c) + g(a,bc) = g(a,b)c \quad (21)$$

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Thus,

$$L = f(g(a,b)c,d) - f(g(a,b),cd) + f(g(a,c),b)d \tag{22}$$

Now, since f is also a 2-cocyle, $d^2(f)(g(a,b),c,d) = 0$, so we have

$$g(a,b)f(c,d) = f(g(a,b),c) - f(g(a,b),cd) + f(g(a,c),b)d \tag{23}$$

This means $L = g(a,b)f(c,d)$. Secondly, evaluating the right hand side of the above differential equation for $f(*,\ g(*,*))$:

$$R = af(b,g(c,d)) - f(ab,g(c,d)) + f(a,g(bc,d)) - f(a,g(b,cd)) + f(a,g(c,b))d \tag{24}$$

By similar argument as above, we obtain $R = f(a,b)g(c,d)$. Finally, we obtain

$$d^3(U)(a,b,c,d) = L - R = g(a,b)f(c,d) - f(a,b)g(c,d) \tag{25}$$

This completes our proof. □

The following Theorem is one of the most important result in formal deformation theory.

Theorem 2.13. *Let* ($A[[t]]$, μ_t) *be a deformation of the algebra A. Then the n-obstructions are Hochschild 3-cocyle for any* $n \geq 2$.

Proof. Applying Lemma 2.12 yields that for $a, b, c, d \in A$:

$$d^3\left(\sum_{i=1}^{n-1}\left(\mu_i(\mu_{n-i}(*,*),*) - \mu_i\left(*,\mu_{n-i}(*,*)\right)\right)\right)(a,b,c,d) \tag{26}$$

$$= \sum_{i=1}^{n-1} d^3\left(\mu_i(\mu_{n-i}(*,*),*) - \mu_i\left(*,\mu_{n-i}(*,*)\right)\right)(a,b,c,d) \tag{27}$$

$$= \sum_{i=1}^{n-1}\left(\mu_{n-i}(a,b)\mu_i(c,d) - \mu_i(a,b)\mu_{n-i}(c,d)\right) \tag{28}$$

$$= \sum_{i=1}^{n-1} d^3\left(\mu_i(\mu_{n-i}(*,*),*) - \mu_i\left(*,\mu_{n-i}(*,*)\right)\right)(a,b,c,d) \tag{29}$$

$$= \sum_{i=1}^{n-1} \mu_{n-i}(a,b)\mu_i(c,d) - \sum_{i=1}^{n-1} \mu_i(a,b)\mu_{n-i}(c,d) = 0 \tag{30}$$

Hence the n-obstruction is a Hochschild 3-cocycle. □

Corollary 2.14. *If the third Hochschild cohomology group* $HH^3(A)$ *is zero, then every Hochschild 2-cocycle of A can be extended to a formal associative deformation of A.*

Proof. Let μ_k be a Hochschild 2-cocycle of A. Set a new multiplication $\mu_t = \mu_0 + \mu_k t^k$ where μ_0 is the original multiplication of the algebra A. Then it is easy to see that ($A[[t]]$, μ_t) is deformation of A because the equation (9) is satisfied. The (k + 1)-obstruction is

$$\sum_{i=1}^{k}\left(\mu_i(\mu_{k-i}(a,b),c) - \mu_i\left(a,\mu_{k-i}(b,c)\right)\right) = d^2(\mu_{k+1}) \tag{31}$$

This obstruction is a 3-cocycle. On the other hand $HH^3(A) = \mathrm{Ker}d^3/\mathrm{Im}d^2 = 0$, there is a Hochschild cochain, denoted by μ_{k+1}, which satisfies (31) for a, b, $c \in A$. Hence $(A[[t]], \mu_t^1 = \mu_0 + \mu_k t^k + \mu_{k+1}t^{k+1})$ is an deformation that extends μ_t. By repeating this process, we obtain formal deformation $\left(A[[t]], \mu_t^\infty = \mu_0 + \sum_{n=k}^{\infty} \mu_n\right)$, and obviously that μ_t^∞ extends μ_t. □

Definition 2.15. Let $\left(A[[t]], \mu_t\right)$ and $\left(A[[t]], \mu_t'\right)$ be two deformations of the algebra A. A formal isomorphism between $\left(A[[t]], \mu_t\right)$ and $\left(A[[t]], \mu_t'\right)$ is a $k[[t]]$-linear map $\phi: \left(A[[t]], \mu_t\right) \to \left(A[[t]], \phi_1\right)$ which is written in the form:

$$\phi(a)=\sum_{n=0}^{\infty}\phi(a)t^n=a+\phi_1(a)t+\phi_2(a)t^2+\cdots,\forall a\in A, \tag{32}$$

where $\phi_1 \in \mathbf{C}^1(A)$, that satisfies

$$\mu_t'\big(\phi(a),\phi(b)\big)=\phi(\mu_t,(a,b)) \tag{33}$$

If such isomorphism exists we say that two deformations $\big(A\big[[t]\big],\mu_t\big)$ and $\big(A\big[[t]\big],\mu_t'\big)$ are equivalent. A formal deformation $\big(A\big[[t]\big],\mu_t\big)$ is said to be trivial if it is isomorphic to the algebra $A\big[[t]\big]$, that is, the multiplication μ_t is pulled back to the usual multiplication of $A\big[[t]\big]$: $\mu_i = 0$ for all $i > 0$.

Expanding both sides of equation (33):

$$\mu_0'(\phi(a),\phi(b))+\mu_1'(\phi(a),\phi(b))t+\cdots=\phi_0\big(\mu_t(a,b)\big)+\phi_1(\mu_t(a,b)t)+\ldots \tag{34}$$

Collecting the coefficients of t^m yields that:

$$\sum_{i+j=m}\phi_i\left(\mu_j(a,b)\right)=\sum_{k+i+j=m}\mu_k'\big(\phi_i(a),\phi_j(b)\big) \tag{35}$$

Proposition 2.16. *Let $\Phi=\sum_{i=0}^{\infty}\phi_i t^i:\big(A\big[[t]\big],\mu_t\big)\to(A[[t]],\mu_t')$ be an isomorphism between two deformations of A. Then $d\phi_1=\mu_1-\mu_1'$, this means the two cocycle are in the same cohomology class. Moreover, $\big(A\big[[t]\big],\mu_t\big)$ is equivalent to a trivial deformation, then μ_1 is a coboundary.*

3. Deformation of *k*-linear categories

Linear categories arises in many areas of mathematics. A prime example is the category of all vector spaces over *k* where its morphisms are *k*-linear maps. Because deformation theory for *k*-linear categories is very similar to the algebra case, we devote this section to present deformations of *k*-linear categories as the first application of deformation theory into the field of category.

Definition 3.1. A category $(M, Ob(M), \mu)$ consists of:

• A class $Ob(M)$ of objects *A*,*B*,*C*,...

• A family of disjoint sets $Hom(A,B)$, one for each pair of $A,B\in Ob(M)$. Write $f:A\to B$ for $f\in Hom(A,B)$, and call f a morphism of *M* with domain *A* and codomain *B*.

• A composition rule, denoted by μ, which assigns to each pair of morphisms $f:A\to B, g:B\to C$ a unique morphism $\mu(f,g)=fg=f\circ g:A\to C$, which is called the composite of f and g. The composition rule satisfies following axioms:

(1) Associativity: If $f:A\to B, g:B\to C, h:C\to D$ then $\mu(f,\mu(g,h))=\mu(\mu(f,g),h)$.

(2) Identity: To each object *B* there exist a morphism $1_B:B\to B$, such that $\mu(1B,f)=f$ and $\mu(g,1_B)=g$ for any $g:A\to B$ and $f:B\to C$.

Definition 3.2. A category $(M, Ob(M), \mu)$ is called a *k*-linear category if:

• For each pair of objects A,B in $Ob(M)$, the set $Hom(A,B)$ is vector space over *k*.

• The composition rule is *k*-linear, that is

$$\mu(af+a'f',g)=a\mu(f,g)+a'\mu(f',g),\text{ and }\mu(f,bg+b'g')=b\mu(f,g)+b'\mu(f,g') \tag{36}$$

for any morphisms $f,f':A\to B, g,g':B\to C$, and scalars $a,a',b,b'\in k$.

There are plenty sources of *k*-linear categories, the one bellows is a prime example of *k*-linear category.

Example 3.3. The category Vect(*k*) which consists vector spaces over *k* as its objects and *k*-linear maps between vector spaces as its morphisms is a *k*-linear category.

Definition 3.4 (Hochschild cohomology for a *k*-linear category *M*)**.** Let $(M, Ob(M), \mu)$ be a *k*-linear category. The group of *n*-Hochschild cochains is

$$\mathbf{C}^n(M)=\prod_{A_0,A_1,\ldots,A_n\in Ob(M)}\hom_k\big(Hom_M(A_0,A_1)\otimes\ldots Hom_M(A_{n-1},A_n);Hom_M(A_0,A_n)\big) \tag{37}$$

and the differential $d^n: \mathbf{C}^n(M) \to \mathbf{C}^{n+1}(M)$ is defined:

$$d^n(f)(a_0, a_1, \ldots, a_n) = a_0 f(a_1, \ldots, a_n) + \sum_{i=0}^{n-1} (-1)^{i+1} f(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^{n+1} f(a_0, \ldots, a_{n-1}) a_n \quad (38)$$

for $f \in \mathbf{C}^n(M)$ and $a_0, a_1, \ldots, a_n$ is a consecutive sequence of morphisms in *M*:

$$A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} A_2 \xrightarrow{a_3} \ldots \xrightarrow{a_n} A_n \quad (39)$$

The *n*-Hochschild cohomology group is

$$\mathrm{HH}^n(M) = \mathrm{H}^n(\mathbf{C}^\bullet(M)) = \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}). \quad (40)$$

Definition 3.5. A formal deformation of the *k*-linear category $(M, Ob(M), \mu)$ is the category $(M_t, Ob(M_t), \mu_t)$ in which:

(1) The class of objects of M_t is the same as of M: $Ob(M_t) = Ob(M)$.

(2) For each pair of objects $A, B \in Ob(M_t)$, the set of morphisms $Hom_{M_t}(A, B)$ is the set of formal power series with coefficients in $Hom_M(A, B)$, that is $Hom_{M_t}(A, B) = Hom_M(A, B)[[t]]$.

(3) The composition rule $\mu_t = \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots = \sum_{i=0}^{\infty} \mu_i t^i$ in which $\mu_0 = \mu$ is the original composition rule in *M*, and $\mu_i \in \mathbf{C}^2(M)$ is Hochschild 2-cochain for $i \geq 1$.

• The deformation $(M_t, Ob(M_t), \mu_t)$ in which $\mu_i = 0$ for $i \geq 1$ ($\mu_t = \mu_0$) is called trivial deformation of *M*.

• If we set $t^2 = 0$ then $(M_t, Ob(M_t), \mu_t = \mu_0 + \mu_1 t)$ is called first-order deformation of *M*.

We enclose this section by the the most important result for deformations of *k*-linear categories as follow.

Theorem 3.6. *First order deformations of k-linear category* $(M, Ob(M), \mu)$ *are classifed by Hochschild cohomology group* $\mathrm{HH}^2(M)$.

Proof. Firstly, assume that $\mu_t = \mu_0 + \mu_1 t$ is a first-order deformation of the category *M* then we prove μ is Hochschild 2-cocyle $\mu_1 \in \mathbf{C}^2(M)$. Indeed, for any three consecutive morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in *M*, the associativity of the composition rule $\mu_t = \mu_0 + \mu_1 t$ holds true $\mu_t(\mu_t(f, g), h) = \mu_t(f, \mu_t(g, h))$. Expanding this equation, we get:

$$\mu_0(\mu_0(f, g), h) + \mu_0(\mu_1(f, g), h)t + \mu_1(\mu_0(f, g)h)t = \mu_0(f, \mu_0(g, h)) + \mu_0(f, \mu_1(g, h))t + \mu_1(f, \mu_0(g, h))t. \quad (41)$$

Collecting the coefficients of *t* in both sides, we obtain that

$$f\mu_1(g, h) - \mu_1(fg, h) + \mu_1(f, gh) - \mu_1(f, g)h = 0 \quad (42)$$

Hence $d^2(\mu_1) = 0$ or μ_1 is a Hochschild 2-cocylce. Secondly, assume that $\mu'_t\, t = \mu_0 + \mu'_1 t$ is another a first-order deformation of the category *M*, and $\Phi = id + \phi_1 t$ is an isomorphism between μ_t and μ'_t. Condition $\mu'(\phi) = \phi(\mu)$ is expanded as:

$$(f + \phi_1(f)t)(g + \phi_1(g)t) + \mu'_1(f + \phi_1(f)t, g + \phi_1(g)t) = fg + \mu_1(f, g)t + \phi_1 \quad (43)$$

Equating the coefficients of *t* in both sides we get:

$$\mu_1(f, g) - \mu'_1(f, g) = f\phi_1(g) - \phi_1(fg) + \phi_1(f)g \quad (44)$$

This means that $\mu_1 - \mu'_1 \in \mathrm{Im}(d^1)$. The proof is completed. □

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Conflict of Interest

The author declares no conflict of interest.

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