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A Note on the Hit Problem for the Steenrod Algebra in Some Degrees

Nhat Duy Thanh Tran[1], **Khac Tin Nguyen**[2*]
[1]*Ho Chi Minh City University of Technology, Vietnam*
[2]*Ho Chi Minh City University of Technology and Education, Vietnam*

[*]Corresponding author. *Email:* tinnk@hcmute.edu.vn

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ABSTRACT

Let $P_k = H^*((RP^\infty)^k)$ be the modulo-2 cohomology algebra of the direct product of k copies of infinite dimensional real projective space RP^∞. Then, P_k is isomorphic to the graded polynomial algebra $F_2[x_1, x_2, \dots, x_k]$ of k variables, in which each x_j is of degree 1, and let GL_k be the general linear group over the prime field F_2 which acts naturally on P_k. Here the cohomology is taken with coefficients in the prime field F_2 of two elements. We study the hit problem, set up by Frank Peterson, of finding a minimal set of generators for the polynomial algebra P_k as a module over the mod-2 Steenrod algebra, A. In this paper, we explicitly compute the hit problem for k = 5 and the degree n=5(2^s-1)+24.2^s with s an arbitrary non-negative integer. Moreover, we get the dimensional results for polynomial algebra in some generic degrees in the case k=6. Note that the main results of this paper have been published online on ArXiv [ArXiv: 2103.04393, Preprint 9 pages, March 7, 2021].

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1. Introduction

Let $P_k = H^*((\mathbb{R}P^\infty)^k)$ denote the modulo-2 cohomology algebra of the direct product of k copies of infinite-dimensional real projective space $\mathbb{R}P^\infty$. This algebra is isomorphic to the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \dots, x_k]$ over the field $\mathbb{F}_2$, where each x_j has degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements.

As the cohomology of a group, P_k is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is determined by the elementary properties of the Steenrod squares Sq^i and the Cartan formula (see Steenrod and Epstein [14]).

A polynomial f in P_k is called a *hit* polynomial if it can be expressed as a finite sum $f = \sum_{u\geqslant 0} Sq^{2^u}(h_u)$ for suitable polynomials h_u. That means f belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ denotes the augmentation ideal in $\mathcal{A}$.

The general linear group GL_k acts naturally on P_k via matrix substitution. Since the actions of $\mathcal{A}$ and GL_k on P_k commute, this induces an action of GL_k on $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

Many authors study the *hit problem* of determination of a minimal set of generators for P_k as a module over the Steenrod algebra, or equivalently, a basis of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. This problem was first studied by Peterson [9], Wood [27], Singer [12], Priddy [11], who showed its relationship to several classical problems in homotopy theory. Then, this problem was investigated by Nam [8], Silverman [13], Wood [27], Sum [15, 17, 18], Tin-Sum [20], Tin [24] and others.

For a positive integer n, by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1\leqslant i\leqslant r}(2^{u_i}-1)$, where $u_i > 0$. This result implies a result of Wood, which originally is a conjecture of Peterson [9].

Theorem 1.1 (Wood [27]). *If $\mu(n) > k$, then $(\mathbb{F}_2\otimes_{\mathcal{A}}P_k)_n = 0$.*

From the above result of Wood, the hit problem is reduced to the case of degree n with $\mu(n) \leqslant k$. For a nonnegative integer d, denote by $(P_k)_d$ the subspace of P_k consisting of all the homogeneous polynomials of degree d in P_k and by $(\mathbb{F}_2\otimes_{\mathcal{A}}P_k)_d$ the subspace of QP_k consisting of all the classes represented by the elements in $(P_k)_d$.

One of the extremely useful tools for computing the hit problem and studying Singer's transfer is the Kameko squaring operation

$$\widetilde{Sq}^0_* := (\widetilde{Sq}^0_*)_{(k,d)} : (\mathbb{F}_2\otimes_{\mathcal{A}}P_k)_{2k+d} \to (\mathbb{F}_2\otimes_{\mathcal{A}}P_k)_d,$$

which is induced by an $\mathbb{F}_2$-linear map $\varphi_k : P_k \to P_k$, given by

$$\varphi_k(x) = \begin{cases} y, & \text{if } x = x_1x_2\dots x_ky^2,\\ 0, & \text{otherwise,}\end{cases}$$

for any monomial $x \in P_k$. The map φ_k is not an $\mathcal{A}$-homomorphism. However, $\varphi_kSq^{2i} = Sq^i\varphi_k$ and $\varphi_kSq^{2i+1} = 0$ for any non-negative integer i.

Theorem 1.2 (Kameko [4]). *Let d be a non-negative integer. If $\mu(2d+k) = k$, then*

$$(\widetilde{Sq}^0_*)_{(k,d)} : (\mathbb{F}_2\otimes_{\mathcal{A}}P_k)_{2d+k} \longrightarrow (\mathbb{F}_2\otimes_{\mathcal{A}}P_k)_d$$

is an isomorphism of GL_k-modules.

Thus, the hit problem is reduced to the case of degree n of the form

$$n = a(2^s-1) + 2^sb,$$

where a, b, m are non-negative intergers such that $0 \leqslant \mu(b) < a \leqslant k$.

Now, the $\mathbb{F}_2$-vector space $\mathbb{F}_2\otimes_{\mathcal{A}}P_k$ was explicitly calculated by Peterson [9] for $k = 1, 2$, by Kameko [4] for $k = 3$ and by Sum [17] for $k = 4$. However, for $k > 4$, it is still unsolved, even in the case of $k = 5$ with the help of computers.

For $a = k-1 = 4$ and $b = 0$, the vector space $(\mathbb{F}_2\otimes_{\mathcal{A}}P_5)_n$ is explicitly computed by Phuc and Sum [10], and by the present author [23] in the case $a = k-1 = 4$, $b \in \{8; 10; 11\}$ with s an arbitrary non-negative integer.

In this paper, we explicitly compute the hit problem for $k = 5$ and the degree $5(2^s-1) + 24.2^s$ with s an arbitrary non-negative integer. Moreover, as a consequence, we get the dimension results for polynomial algebra in some generic degrees in the case $k = 6$.

2. Preliminaries

In this section, we recall some needed information from Kameko [4], Singer [12] and Sum [15], which will be used in the next section.

Notation 2.1. We denote $\mathbb{N}_k = \{1, 2, \dots, k\}$ and

$$X_{\mathbb{J}} = X_{\{j_1,j_2,\dots,j_s\}} = \prod_{j\in\mathbb{N}_k\setminus\mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \dots, j_s\} \subset \mathbb{N}_k,$$

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \dots$, for $\alpha_i(a) = 0$ or 1 with $i \geqslant 0$.

Let $x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} \in P_k$. Denote $\nu_j(x) = a_j, 1 \leqslant j \leqslant k$. Set

$$\mathbb{J}_t(x) = \{j \in \mathbb{N}_k : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geqslant 0$. Then, we have $x = \prod_{t \geqslant 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.2. For a monomial x in P_k, define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \dots, \omega_i(x), \dots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \dots, \nu_k(x)),$$

where $\omega_i(x) = \sum_{1 \leqslant j \leqslant k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)}$, $i \geqslant 1$. The sequence $\omega(x)$ is called the weight vector of x.

Let $\omega = (\omega_1, \omega_2, \dots, \omega_i, \dots)$ be a sequence of non-negative integers. The sequence ω is called the weight vector if $\omega_i = 0$ for $i \gg 0$.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

For a weight vector ω, we define $\deg \omega = \sum_{i>0} 2^{i-1}\omega_i$. Denote by $P_k(\omega)$ the subspace of P_k spanned by all monomials y such that $\deg y = \deg \omega$, $\omega(y) \leqslant \omega$, and by $P_k^-(\omega)$ the subspace of P_k spanned by all monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$.

Definition 2.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k.

i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+ P_k$. If $f \equiv 0$ then f is called *hit*.

ii) $f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+ P_k + P_k^-(\omega)$.

Obviously, the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$. Then, we have

$$QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+ P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

For a polynomial $f \in P_k$, we denote by $[f]$ the class in QP_k represented by f. If ω is a weight vector, then denote by $[f]_\omega$ the class represented by f. Denote by $|S|$ the cardinal of a set S.

Definition 2.4. Let x, y be monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds:

i) $\omega(x) < \omega(y)$;

ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.5. A monomial x is said to be inadmissible if there exist monomials $y_1, y_2, \dots, y_m$ such that $y_t < x$ for $t = 1, 2, \dots, m$ and $x - \sum_{t=1}^m y_t \in \mathcal{A}^+ P_k$.

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n.

Definition 2.6. Let u be a monomial in $\mathcal{P}_n$. We say u is strictly inadmissible if there exist monomials $v_1, v_2, \dots, v_m$ such that $v_j < u$, for $j = 1, 2, \dots, m$ and $u = \sum_{j=1}^m v_j + \sum_{i=1}^{2^s-1} Sq^i(f_i)$ with $s = \max\{k : \omega_k(u) > 0\}$, $f_i \in \mathcal{P}_n$.

Observe that if u is strictly inadmissible monomial, then it is inadmissible monomial, as defined by the definitions 2.5, and 2.6. In general, the inverse is not true.

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Theorem 2.7 (Kameko [4], Sum [17]). *Let u, v, w be monomials in $\mathcal{P}_n$ such that $\omega_t(u) = 0$ for $t > k > 0$, $\omega_r(w) \neq 0$ and $\omega_t(w) = 0$ for $t > r > 0$. Then,*

(i) uw^{2^k} *is inadmissible if w is inadmissible.*

(ii) wv^{2^r} *is strictly inadmissible if w is strictly inadmissible.*

Definition 2.8. Let $z = x_1^{d_1}x_2^{d_2}\ldots x_n^{d_n}$ in $\mathcal{P}_n$. The monomial z is called a spike if $d_j = 2^{t_j} - 1$ for t_j a non-negative integer and $j = 1, 2, \ldots, n$. Moreover, z is called the minimal spike, if it is a spike such that $t_1 > t_2 > \ldots > t_{r-1} \geqslant t_r > 0$ and $t_j = 0$ for $j > r$.

The following is a Singer's criterion on the hit monomials in $\mathcal{P}_n$.

Theorem 2.9 (**Singer** [12]). *Assume that z is the minimal spike of degree d in $\mathcal{P}_n$, and $u \in (\mathcal{P}_n)_d$ satisfying the condition $\mu(d) \leqslant n$. If $\omega(u) < \omega(z)$, then u is hit.*

Now, we recall some notations and definitions in [17], which will be used in the next sections. We set

$$P_k^0 = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \ : \ a_1a_2\ldots a_k = 0\}\rangle,$$
$$P_k^+ = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \ : \ a_1a_2\ldots a_k > 0\}\rangle.$$

It is easy to see that P_k^0 and P_k^+ are the $\mathcal{A}$-submodules of P_k. Furthermore, we have the following.

Proposition 2.10. *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k = QP_k^0 \oplus QP_k^+$. Here $QP_k^0 = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^0$ and $QP_k^+ = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+$.*

Definition 2.11. For any $1 \leqslant i \leqslant k$, define the homomorphism $f_i : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ x_{j+1}, & \text{if } i \leqslant j < k. \end{cases}$$

Then, f_i is a homomorphism of $\mathcal{A}$-modules.

For a subset $B \subset P_k$, we denote $[B] = \{[f] : f \in B\}$. Obviously, we have

Proposition 2.12. *It is easy to see that if B is a minimal set of generators for $\mathcal{A}$-module P_{k-1} in degree n, then $f(B) = \bigcup_{i=1}^k f_i(B)$ is a minimal set of generators for $\mathcal{A}$-module P_k^0 in degree n.*

From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k, $B_k^0(n) = B_k(n) \cap P_k^0$, $B_k^+(n) = B_k(n) \cap P_k^+$. For a weight vector ω of degree n, we set $B_k(\omega) = B_k(n) \cap P_k(\omega)$, $B_k^+(\omega) = B_k^+(n) \cap P_k(\omega)$.

Then, $[B_k(\omega)]_\omega$ and $[B_k^+(\omega)]_\omega$, are respectively the basses of the $\mathbb{F}_2$-vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$.

3. The Main Results

We first recall a result in [20] the following: Let d be an arbitrary non-negative integer. Set

$$t(k, d) = \max\{0, k - \alpha(d + k) - \zeta(d + k)\},$$

where $\zeta(n)$ the greatest integer u such that n is divisible by 2^u, that means $n = 2^{\zeta(n)}m$, with m an odd integer.

Theorem 3.1 (Tin-Sum [20]). *Let d be an arbitrary non-negative integer. Then*

$$(\widetilde{Sq}^0_*)^{s-t} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{k(2^s-1)+2^s d} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{k(2^t-1)+2^t d}$$

is an isomorphism of GL_k-modules for every $s \geqslant t$ if and only if $t \geqslant t(k,d)$.

It is easy to check that for $k = 5$ and $d = 53$ then

$$t(k,d) = \max\{0, k - \alpha(d+k) - \zeta(d+k)\} = 0.$$

Using the above theorem, we get $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^t-1)+2^t 53} \cong (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^0-1)+2^0 53}$ for all $t \geqslant 0$. Therefore, we need only to study $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^s-1)+24.2^s}$ for $s = 0; 1$.

The case s=0. Denote by $\omega_{(1)} = (4,4,3)$, $\omega_{(2)} = (4,4,1,1)$, $\omega_{(3)} = (4,2,2,1)$, and $\omega_{(4)} = (4,2,4)$. We give a direct summand decomposition of the $\mathbb{F}_2$-vector spaces $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^0-1)+24.2^0}$ as follows:

Theorem 3.2. *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces*

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{24} = (QP_5^0)_{24} \oplus QP_5^+(\omega_{(1)}) \oplus QP_5^+(\omega_{(2)}) \oplus QP_5^+(\omega_{(3)}) \oplus QP_5^+(\omega_{(4)}).$$

Proof. By Proposition 2.10, it suffices to show that $(QP_5^+)_{24} = \bigoplus_{d=1}^4 QP_5^+(\omega_{(d)})$. Observe that $z = x_1^{15}x_2^7x_3x_4$ is the minimal spike of degree twenty-four in $\mathcal{P}_5$, with $\omega(z) = (4,2,2,1)$. Assume that u is an admissible monomial of degree twenty-four in $\mathcal{P}_5^+$. Using Theorem 2.9, we obtain $\omega_1(u) \geqslant \omega_1(z) = 4$. Since the degree of u is even, one gets $\omega_1(u) = 4$. Hence, $u = X_{\{j\}}v^2$, with v an admissible monomial of degree ten in $\mathcal{P}_5$, and $1 \leqslant j \leqslant 5$. Using the result in Tin [21], one has $\omega(v) = (4,3)$ or $\omega(v) = (4,1,1)$, or $\omega(v) = (2,2,1)$ or $\omega(v) = (2,4)$. Therefore, $\omega(u) = \omega_{(d)}$ for all $d = 1,2,3,4$. Hence, it shows that

$$(QP_5^+)_{24} = QP_5^+(\omega_{(1)}) \oplus QP_5^+(\omega_{(2)}) \oplus QP_5^+(\omega_{(3)}) \oplus QP_5^+(\omega_{(4)}).$$

The theorem is proved. □

Recall that $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_4)_{10}$ is an $\mathbb{F}_2$-vector space of dimension 70 with a basis consisting of all the classes represented by the monomials w_j, $1 \leqslant j \leqslant 70$. Consequently, $|B_4(10)| = 70$, (see Sum [17]).

Since $\mu(24) = 4$, Theorem 1.2 implies that the squaring operation

$$\widetilde{Sq}^0_* : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_4)_{24} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_4)_{10}$$

is an isomorphism of GL_4-module. Hence, $|B_4(24)| = |B_4(10)| = 70$, and therefore the set

$$[B_4(24)] = \{[\phi_4(w_j)] : w_j \in B_4(10)\}$$

is a basis of the $\mathbb{F}_2$-vector space $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_4)_{24}$, where $\phi_4(u) = x_1x_2x_3x_4u^2$.

Using Proposition 2.12, we obtain

$$[B_5^0(24)] = \left\{ [a_t] : \ a_t \in \bigcup_{i=1}^5 f_i(B_4(24)), 1 \leqslant t \leqslant 350 \right\}$$

is a basis of the $\mathbb{F}_2$-vector space $(QP_5^0)_{24}$. Consequently,

$$\dim(QP_5^0)_{24} = |\bigcup_{i=1}^5 f_i(B_4(24))| = 350.$$

Remark 3.3. We recall a result in Mothebe-Kaelo-Ramatebele [7] as follows.

Set $\mathcal{M}_{(k,r)} = \{J = (j_1, j_2, \ldots, j_r) : 1 \leqslant j_1 < \ldots < j_r \leqslant k\}$, $1 \leqslant r < k$. For $J \in \mathcal{M}_{(k,r)}$, define the homomorphism $f_J : P_r \to P_k$ of algebras by substituting $f_J(x_t) = x_{j_t}$ with $1 \leqslant t \leqslant r$. Then, f_J is a monomorphism of $\mathcal{A}$-modules. We have a direct summand decomposition of the $\mathbb{F}_2$-vector subspaces:

$$QP_k^0 = \bigoplus_{1\leqslant r\leqslant k-1} \bigoplus_{J\in\mathcal{M}_{(k,r)}} (Qf_J(P_r^+)),$$

where, $Qf_J(P_r^+) = \mathbb{F}_2 \otimes_{\mathcal{A}} f_J(P_r^+)$.

In degree n, we have $\dim(Qf_J(P_r^+))_n = \dim(QP_r^+)_n$ and $|\mathcal{M}_{(k,r)}| = \binom{k}{r}$. Hence, combining with Theorem 1.1 we get

$$\dim(QP_k^0)_n = \sum_{\mu(n)\leqslant r\leqslant k-1} \binom{k}{r} \dim(QP_r^+)_n.$$

And therefore, using the results in Sum [17], one gets $\dim(QP_5^0)_{24} = 350$.

Next, we explicitly determine the $\mathbb{F}_2$-vector spaces $QP_5^+(\omega_{(d)}), 1 \leqslant d \leqslant 4$, by showing an admissible monomial basis of $P_5(\omega_{(d)})$ for $d \in \{1, 2, 3, 4\}$. Specifically, we obtain the following result:

Theorem 3.4.

$$\dim(QP_5(\omega_{(d)})) = \begin{cases} 75, & \textit{if } \ d = 1, \\ 145, & \textit{if } \ d = 2, \\ 390, & \textit{if } \ d = 3, \\ 1, & \textit{if } \ d = 4. \end{cases}$$

Proof. Consider the weight vector $\omega = \omega_{(4)} = (4, 2, 4)$. Suppose x is an admissible monomial in $\mathcal{P}_5$ such that $\omega(x) = (4, 2, 4)$, then $x = X_j y^2$, where $y \in B_5(10)$, and $1 \leqslant j \leqslant 5$. We set

$$\mathcal{D}_5^1 := \{x_i x_j x_k x_\ell . y^2 : \omega(y) = (2, 4),\ 1 \leqslant i < j < k < \ell \leqslant 5\} \ \cap \ P_5^+.$$

It is easy to see that $\text{span}\{\mathcal{D}_5^1\} = P_5^+(\omega_{(4)})$, and if $u \in \mathcal{D}_5^1$ then $u = x_i^3 x_j^6 x_k^5 x_\ell^5 x_m^5$, or $u = x_i^2 x_j^7 x_k^5 x_\ell^5 x_m^5$, or $u = x_i^3 x_j^7 x_k^4 x_\ell^5 x_m^5$. Here (i, j, k, ℓ, m) is an arbitrary permutation of $(1, 2, 3, 4, 5)$, and $i < k; \ell; m$.

By means of direct computation using the Cartan formula, we have established that only the monomial $x_1^3 x_2^5 x_3^5 x_4^5 x_5^6$ is admissible, whereas the others in $\mathcal{D}_5^1$ are inadmissible. Moreover, the set $\{[x_1^3 x_2^5 x_3^5 x_4^5 x_5^6]\}$ is linearly independent in QP_5. Hence, the set $\{[x_1^3 x_2^5 x_3^5 x_4^5 x_5^6]\}$ is a basis of the $\mathbb{F}_2$-vector space $QP_5^+(\omega_{(4)})$. That means, $\dim QP_5^+(\omega_{(4)}) = 1$.

The proofs for the remaining weight vectors in the above theorem follow a similar approach, explicitly determining all admissible monomials of $P_5(\omega_{(d)})$ for $d \in \{1, 2, 3\}$. It is worth noting that the calculations in this proof are lengthy and highly technical. Readers seeking more details about these admissible monomials may contact the authors of this article via email. □

From the above results, we get the corollary following.

Corollary 3.5. *There exist exactly* 961 *admissible monomials of degree twenty-four in* P_5. *Consequently,* $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{24} = 961$.

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The case s=1. Since Kameko's homomorphism $(\widetilde{Sq}^0_*)_{(5,24)}$ is an epimorphism, we have

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{53} \cong \mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,24)} \bigoplus \mathrm{Im}(\widetilde{Sq}^0_*)_{(5,24)}.$$

Therefore, we immediately have the following result, which is a consequence of Corollary 3.5.

Theorem 3.6. *$Im(\widetilde{Sq}^0_*)_{(5,24)}$ is isomorphic to a subspace of $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{53}$ generated by all the classes represented by the admissible monomials of the form $x_1x_2\ldots x_5u^2$, for every $u \in B_5(24)$. Consequently, $\dim(Im(\widetilde{Sq}^0_*)_{(5,24)}) = 961$.*

Next, we explicitly determine $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,24)}$ by giving a direct summand decomposition of the $\mathbb{F}_2$-vector spaces as follows.

Theorem 3.7. *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces*

$$Ker(\widetilde{Sq}^0_*)_{(5,24)} = (QP_5^0)_{53} \oplus QP_5^+(3,3,3,2,1).$$

Proof. Suppose that u is an admissible monomial of degree fifty-three in $\mathcal{P}_5$ such that $[u]$ is in $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,24)}$. Observe that $w = x_1^{31}x_2^{15}x_3^7$ is the minimal spike of degree fifty-three in $\mathcal{P}_5$, with $\omega(w) = (3,3,3,2,1)$. Using Theorem 2.9, we obtain $\omega_1(u) \geqslant \omega_1(w) = 3$. Since the degree of u is odd, one gets either $\omega_1(3) = 3$, or $\omega_1(u) = 5$.

If $\omega_1(u) = 5$ then $u = X_\emptyset v^2$, where v is a monomial of degree twenty-four in $\mathcal{P}_5$. Since u is admissible, Theorem 2.7 implies that v is also admissible, and $[v] \neq 0$. Hence, $[v] = (\widetilde{Sq}^0_*)_{(5,24)}([u]) \neq 0$, which contradicts the fact that $[x]$ belongs to $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,24)}$.

Hence, $\omega_1(u) = 3$ then $u = X_{\{i,j\}}v^2$, where v is an admissible monomial of degree twenty-five in $\mathcal{P}_5$, and $1 \leqslant i < j \leqslant 5$. Using the result in Sum [18], one has $\omega(v) = (3,3,2,1)$.

So, $\omega(u) = (3,3,3,2,1)$. And therefore, $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,24)} = (QP_5^0)_{53} \oplus QP_5^+(\omega(u))$. The theorem is proved. □

Remark 3.8. From the result in [7] we have

$$\dim(QP_k^0)_{53} = \sum_{\mu(53)\leqslant r\leqslant 4} \binom{5}{r} \dim(QP_r^+)_{53}.$$

Since $\mu(53) = 3$, $\dim(QP_3^+)_{53} = 8$ (see Kameko [4]) and $\dim(QP_4^+)_{53} = 88$ (see Sum [17]), one gets

$$\dim(QP_5^0)_{53} = \binom{5}{3}.\dim(QP_3^+)_{53} + \binom{5}{4}.\dim(QP_4^+)_{53} = 520.$$

And therefore the set $\{[b_t] :\ b_t \in \bigcup_{i=1}^5 f_i(B_4(53)), 1 \leqslant t \leqslant 520\}$ is a basis of the $\mathbb{F}_2$-vector space $(QP_5^0)_{53}$.

We denote by $\mathbb{M}^d(n) = \{[\bigcup_{i=1}^5 x_i^{2^d-1} f_i(x)] : x \in B_{k-1}(n - 2^d + 1)\}$ and set $\mathbb{M} = \mathrm{span}\{[u] :\ u \in \bigcup_{d=1}^5 \mathbb{M}^d(53)$ and $\omega(u) = (3,3,3,2,1)\}$. Hence, one gets the theorem following.

Theorem 3.9. *The following statements are true:*
i) $\mathbb{M}$ *is the* $\mathbb{F}_2$*-vector subspaces of* $QP_5^+(\widetilde{\omega})$, *where* $\widetilde{\omega} = (3,3,3,2,1)$.
ii) *Assume that* $QP_5^+(\widetilde{\omega}) = \mathbb{M} \oplus \mathbb{N}$. *Then,* $\dim(\mathbb{M}) = 389$, *and* $\dim(\mathbb{N}) = 851$.

Proof. The proof of Part (i) of the above theorem is straightforward. Next, we prove Part (ii) of the theorem. Using the same arguments as in the proof of the previous theorem, we determine $QP_5^+(\widetilde{\omega})$ by explicitly identifying all admissible monomials in $P_5^+(\widetilde{\omega})$. However, providing a complete admissible monomial basis for these subspaces would be excessively long and computationally complex. Readers seeking more details about these admissible monomials may contact the authors via email. The following is a sketch of the proof, assisted by computer computations:

Using the results in Sum [17], one has

$$|B_4(53 - 2^d + 1)| = \begin{cases} 70, & \text{if } d = 1, \\ 155, & \text{if } d = 2, \\ 164, & \text{if } d = 3, \\ 192, & \text{if } d = 4, \\ 116, & \text{if } d = 5. \end{cases}$$

An easy computation shows that

$$\left| \bigcup_{d=1}^{5} \bigcup_{i=1}^{5} x_i^{2^d - 1} f_i(B_4(53 - 2^d + 1)) \right| = 389.$$

We will denote by $\mathcal{D}_5^2(\widetilde{\omega})$ the set of classes represented by the admissible monomials of the vector space $QP_5^+(\widetilde{\omega}) \setminus \mathbb{M}$. Consider the set

$$B_5^+(\widetilde{\omega}) := \{X_{\{i,j\}}.p^2 : p \in B_5(25), 1 \leqslant i < j \leqslant 5\} \cap \mathcal{P}_5^+.$$

Using Theorem 2.7, it shows that if u is an admissible monomial of degree 53 in $\mathcal{P}_5^+$ such that $[u]$ does not belong to $\mathrm{Im}(\widetilde{Sq}_*^0)_{(5,24)}$, then $u \in B_5^+(\widetilde{\omega})$.

By observing that each monomial $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}$ corresponds to a series of numbers of the type $(a_1; a_2; a_3; a_4; a_5)$, we set up an algorithm implemented in Microsoft Excel software to eliminate the inadmissible monomials in $B_5^+(\widetilde{\omega})$. By direct calculations, using Theorem 2.7, one gets $|B_5^+(\widetilde{\omega})| = 1240$. And therefore, we obtain $\dim(\mathbb{N}) = |\mathcal{D}_5^2(\widetilde{\omega})| = 851$. The theorem has been established. □

Considering the homomorphism $\psi : P_5 \to P_5$ is an $\mathbb{F}_2$-homomorphism determined by $\psi(u) = \prod_{j=1}^{5} x_j u^2$, for $u \in P_5$. Using the Theorem 3.1, we obtain the following result:

Corollary 3.10. *The set* $\{[x] : x \in \psi^{s-1}(B_5(53))\}$ *is a basis of the* $\mathbb{F}_2$*-vector space* $(QP_5)_{5(2^s-1)+24.2^s}$ *for any* $s > 0$. *Consequently,* $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^s-1)+24.2^s} = 2201$, *for each* $s \geqslant 1$.

It is well known that after explicitly determining QP_4, Sum [17] derived an inductive formula in terms of n for the dimension of the vector space $(QP_n)_d$, where d represents a general degree (see Theorem 1.3, in Nguyen Sum [Adv. Math. 274 (2015) 432-489]). Since $\mu(53) = 3 = \alpha(53 + \mu(53))$, using the result in Sum [17], one obtains

$$|B_6(5(2^u - 1) + 2^u 53)| = (2^6 - 1)|B_5(53)|, \text{ for any integer } u \geqslant k - 1 = 5.$$

We obtain the following.

Corollary 3.11. *There exist exactly* 138663 *admissible monomials of degree* $m_0 = 5(2^u - 1) + 53.2^u$ *in* P_6*, for any* $u > 4$*. Consequently,* $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_6)_{m_0} = 138663$.

On the other hand, we also see that $|B_5(25)| = 1240$ (see [18]) and $|B_5(19)| = 905$ (see [21]). Since $\mu(25) = 3 = \alpha(25 + \mu(25))$, using the result in Sum [17], one gets

$$|B_6(5(2^v - 1) + 2^v 25)| = (2^6 - 1)|B_5(25)|, \text{ for any integer } v \geqslant k - 1 = 5.$$

So, we obtain the corollary following.

Corollary 3.12. *There exist exactly* 78120 *admissible monomials of degree* $m_1 = 5(2^v - 1) + 25.2^v$ *in* P_6*, for any* $v > 4$*. Consequently,* $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_6)_{m_1} = 78120$.

Similarly, it is easy to check that $\mu(19) = 3 = \alpha(\mu(19) + 19)$, using the result in Sum [17] we get the following.

Corollary 3.13. *There exist exactly* 57015 *admissible monomials of degree* $m_2 = 5(2^r - 1) + 19.2^r$ *in* P_6*, for any* $r \geqslant 5$*. Consequently,* $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_6)_{m_2} = 57015$.

Remark 3.14. One of the major applications of hit problem is in studying a homomorphism introduced by Singer. In [12], Singer defined the algebraic transfer, which is a homomorphism

$$\varphi_k : \mathrm{Tor}^{\mathcal{A}}_{k,k+d}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)^{GL_k}_d$$

from the homology of the Steenrod algebra to the subspace of $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_d$ consisting of all the GL_k-invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $\mathrm{Tor}^{\mathcal{A}}_{k,k+d}(\mathbb{F}_2, \mathbb{F}_2)$. The hit problem and the algebraic transfer have been studied by many authors (see Boardman [1], Nam [8], Phuc and Sum [10], Sum [15, 16, 17], Sum and Tín [19] and others).

Singer showed in [12] that φ_k is an isomorphism for $k = 1, 2$. Boardman showed in [1] that φ_3 is also an isomorphism. However, for any $k \geqslant 4$, φ_k is not a monomorphism in infinitely many degrees (see Singer [12], Hung [3].) Singer made the following conjecture.

Conjecture 3.15 (Singer [12]). *The algebraic transfer* φ_k *is an epimorphism for any* $k \geqslant 0$.

The conjecture is true for $k \leqslant 3$. However, for $k > 3$, it is still open. We hope that the conjecture is also true in this case. We make the following conjecture on Singer's conjecture for the algebraic transfer in the case $k = 5$ and at the above degrees.

Conjecture 3.16. *Singer's conjecture is true for* $k = 5$ *and the degree* $5(2^s - 1) + 24.2^s$*, with* s *an arbitrary non-negative integer.*

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Nhat Duy Thanh Tran received Master of Algebra and Number theory at The University of Science, Viet Nam National University Ho Chi Minh City in 2011. He is currently a lecturer at Ho Chi Minh City University of Technology (HUTECH), Vietnam from 2022. His research interests include matrix, commutative algebra, algebraic topology.

Email: tnd.thanh@hutech.edu.vn; trannhatduythanh@gmail.com. ORCID: https://orcid.org/0009-0006-9675-7148

Khac Tin Nguyen received a Ph.D. in mathematics from Quy Nhon University, Vietnam. From 2009 to now, he has been a lecturer at the Faculty of Applied Sciences, Ho Chi Minh City University of Technology and Education, Ho Chi Minh City, Vietnam. His research fields include Peterson hit problem, algebraic transfer, and cohomology of Steenrod algebra. Email: tinnk@hcmute.edu.vn. ORCID: https://orcid.org/0000-0001-6107-9769