

A UNIFIED THIRD-ORDER SHEAR DEFORMATION THEORY FOR STATIC ANALYSIS OF LAMINATED COMPOSITE BEAMS

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ABSTRACT

A unified higher-order shear deformation theory for static analysis of laminated composite beams is proposed in this paper. The theory is based on a higher-order shear deformation beam theory in which a general shear function is proposed. The characteristic equations are derived from Lagrange's equations and then Ritz method is used to determine the stiffness matrix. As for static analysis, the potential kinetic energy in the Lagrange's equation is ignored. The shape functions for Ritz method approximation of displacement variables are selected to satisfy the boundary conditions. Numerical results are compared to those from previous works and are used to investigate the effects of fiber orientation, span-to-thickness ratio and boundary conditions on deflection and stresses of laminated composite beams.

Keywords: Beam theory; Composite materials; Ritz method; Static analysis; Trigonometric functions.

1. INTRODUCTION

Composite materials are engineering materials which consist of two or more material phases whose mechanical performance and properties are designed to be superior to those of the constituents. Laminated composite (LC) beams have been applied in many engineering fields such as aerospace, mechanical, construction. Many studies have been investigated to accurately predict static, buckling and vibration of the LC beams, only some representative previous works are herein cited. Chandrashekhara and Bangera [1] have studied free vibration of LC beams using finite element method (FEM). Khdeir and Reddy [2] analysed free vibration and stability of LC beams with different boundary conditions (BCs). Marur and Kant [3] have analysed free vibration of LC beams using higher-order shear deformation theory (HSDT) and FEM. Matsunaga [4] have utilised the HSDT to study the free vibration and stability of LC beams. Aydogdu [5] investigated free vibration of LC beams under various BCs by using Ritz method. Vo

et al. [6] have researched on static behaviors of LC beams using different refined HSDTs. These authors also applied this theory in studying the vibration and stability of LC beams [7]. In addition, Vo et al. [8] used a refined shear deformation theory to study the free vibration of rectangular-sectioned LC beams under axial loads. Soldatos et al. [9] have investigated orthotropic LC beams using a unified shear transverse theory. Shi et al. [8] have analysed the vibration of LC beams based on a HSDT and FEM. Vidal et al. [10] have proposed a finite element solution to analyse rectangular-sectioned LC beams. Wanji et al. [11] have analysed the bending of LC beams under mechanical loads using first-order shear deformation theory and modified couple stress model. Wu et al. [12] have assessed several displacement-based theories for the vibration and stability analysis of LC sandwich beams. Nguyen et al. [13] have analysed the free vibration of LC beams based on the HSDT and the Ritz method. Yogesh [14] has analysed the free vibration of LC beams with various BCs. Aguiar et al. [15] have

presented the mixed and displacement-based models for static analysis of LC beams with different cross-sections. Khdeir et al. [2, 16] have proposed an exact solution for the bending and free vibration responses of thin and thick cross-fly LC beams. Murthy et al. [17] have presented a refined higher-order finite element for asymmetric LC beams. Zenkour [18] has analysed the bending of LC sandwich beams using transverse shear and normal deformation theory. Chakraborty et al. [19] have analysed free vibration and wave propagation in asymmetric LC beams using FEM. Mantari et al. [20] have studied the free vibration and buckling of LC beams via hybrid Ritz solution with different BCs. Karama et al. [21] have researched the mechanical behaviors of LC beams by using new multi-layered LC structures model with transverse shear stress continuities. Arya et al. [22] have presented a zigzag model for LC beams. Akavci et al. [23] have analysed the buckling and free vibration of LC beams by using two new hyperbolic shear deformation theories. A literature review shows that many type of research have investigated bending behaviors of LC beams using the HSDTs. In practice, this theory overcomes the adverse of the first-order shear deformation theory and predict more accurate; however, practically its accuracy depends on a suitable choice of shear functions.

The objective of this paper is to present a unified shear deformation theory for static analysis of LC beams. The governing equations are derived from Lagrange's equations and then the Ritz method is used to solve the characteristic problem. Numerical results are compared to those from previous works and to investigate the effects of fiber angle, span-to-thickness ratio and BCs on the deflection and stresses of the LC beams.

2. THEORETICAL FORMULATION

Consider a LC beam as Fig. 1 with length L and rectangle section $b \times h$. The beam is composed of n layers made of orthotropic materials.

2.1 Kinematic, strain and stress

The displacement field of a common higher-order shear deformation beam theory is given by ([24]):

$$u_1(x, z, t) = u(x, t) - zw_{,x} + f(z)\theta(x, t) \quad (1a)$$

$$u_3(x, z, t) = w(x, t) \quad (1b)$$

where u , w are axial and transverse displacements at the mid-axis of the beam, respectively; θ is rotation; $f(z)$ is a shear function. The comma in subscript is used to indicate the differentiation with the index that follows.

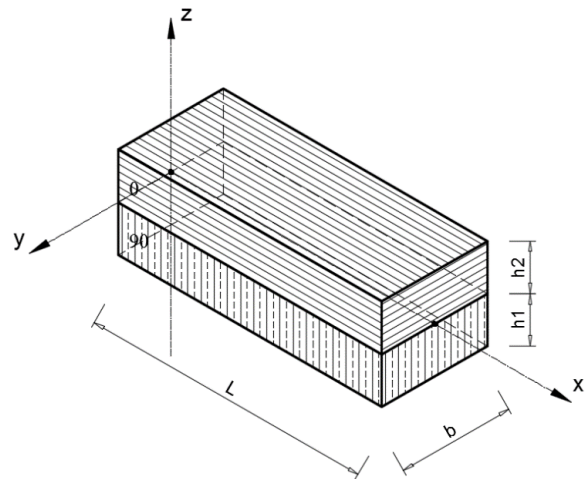


Fig.1 Geometry of a LC beam

The non-zero strains associated to the displacements in Eq. (1) are expressed by:

$$\varepsilon_x(x, z, t) = \varepsilon_x^0 + z\kappa_x^b + f\kappa_x^s \quad (2a)$$

$$\gamma_{xz}(x, z, t) = g(z)\gamma_{xz}^0 \quad (2b)$$

where $g(z) = f_{,z}$ that satisfies the traction-free boundary conditions on the top and bottom faces of the beam, i.e. $g\left(z = \pm \frac{h}{2}\right) = 0$. The

strains ε_x^0 , γ_{xz}^0 , κ_x^b , κ_x^s associated to the displacements in Eq. (1) are given by:

$$\varepsilon_x^0 = u_{,x} \quad (3a)$$

$$\gamma_{xz}^0 = \theta \quad (3b)$$

$$\kappa_x^b = -w_{,xx} \quad (3c)$$

$$\kappa_x^s = \theta_{,x} \quad (3d)$$

The elastic constitutive equations of laminated composite beams at the k^{th} – layer are defined as follows:

$$\sigma_x^{(k)}(x, z, t) = \bar{Q}_{11}^{(k)} \varepsilon_{xx} \quad (4a)$$

$$\sigma_{xz}^{(k)}(x, z, t) = \bar{Q}_{55}^{(k)} \gamma_{xz} \quad (4b)$$

where $\bar{Q}_{11}, \bar{Q}_{55}$ are reduced stiffness components (see [25] for details).

2.2 Energy functional

The total energy of the composite beam is composed of strain energy and work of external loads:

$$\Pi = U + V \quad (5)$$

where U, V are strain energy and work done by external force, respectively. The strain energy of the beam is given by:

$$U = \frac{1}{2} \int_V (\sigma_{xx} \varepsilon_{xx} + \sigma_{xz} \gamma_{xz}) dV$$

$$= \frac{1}{2} \int_0^L \left[\begin{aligned} &A(u_{,x})^2 - 2Bu_{,x}w_{,xx} + D(w_{,xx})^2 \\ &+ 2B^s u_{,x} \theta_{,x} - 2D^s w_{,xx} \theta_{,x} \\ &+ H^s (\theta_{,x})^2 + A^s \theta^2 \end{aligned} \right] dx \quad (6)$$

where (A, B, D, B^s, D^s, H^s) are stiffnesses of the beam defined by:

$$(A, B, D, B^s, D^s, H^s) = \sum_{k=1}^n \left(\int_{z_k}^{z_{k+1}} (1, z, z^2, f, zf, f^2) \bar{Q}_{11}^{(k)} b dz \right) \quad (7a)$$

$$A^s = \sum_{k=1}^n \left(\int_{z_k}^{z_{k+1}} g^2 \bar{Q}_{55}^{(k)} b dz \right) \quad (7b)$$

The work of transverse load q of the beam is defined by:

$$V = - \int_0^L q w b dx \quad (8)$$

Substituting Eqs. (6) and (8) into Eq. (5) leads to:

$$\Pi = \frac{1}{2} \int_0^L \left[\begin{aligned} &A(u_{,x})^2 - 2Bu_{,x}w_{,xx} + D(w_{,xx})^2 \\ &+ 2B^s u_{,x} \theta_{,x} - 2D^s w_{,xx} \theta_{,x} \\ &+ H^s (\theta_{,x})^2 + A^s \theta^2 \end{aligned} \right] dx \quad (9)$$

$$- \int_0^L q w b dx$$

2.3 Governing equations

In order to derive characteristic equations of the beam, the displacement variables are approximated with respect to the Ritz method as follows:

$$u(x) = \sum_{j=1}^m \psi_j(x) u_j \quad (10a)$$

$$w(x) = \sum_{j=1}^m \varphi_j(x) w_j \quad (10b)$$

$$\theta(x) = \sum_{j=1}^m \psi_j(x) \theta_j \quad (10c)$$

where φ, ψ are shape functions which are chosen to satisfy the essential boundary conditions (S-S: simply supported beams, C-C: clamped-clamped beams) as follows:

$$\varphi(x) = \sin \frac{j\pi x}{L}, \quad \psi(x) = \cos \frac{j\pi x}{L} \quad \text{for S-S}$$

$$\text{and} \quad \varphi(x) = \sin^2 \frac{j\pi x}{L}, \quad \psi(x) = \sin \frac{2j\pi x}{L}$$

for C-C. Substituting Eqs. (10) into Eq. (9)

and using Lagrange's equations $\frac{\partial \Pi}{\partial d_j} = 0$ with

$d_j = \{u_j, w_j, \theta_j\}$ leads to:

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} \\ {}^T \mathbf{K}^{12} & \mathbf{K}^{22} & \mathbf{K}^{23} \\ {}^T \mathbf{K}^{13} & {}^T \mathbf{K}^{23} & \mathbf{K}^{33} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{w} \\ \boldsymbol{\theta} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{F} \\ \mathbf{0} \end{Bmatrix} \quad (11)$$

where the components of stiffness and mass matrices are defined by:

$$K_{ij}^{11} = A \int_0^L \psi_{i,x} \psi_{j,x} dx \quad (12a)$$

$$K_{ij}^{12} = -B \int_0^L \psi_{i,x} \varphi_{j,xx} dx \quad (12b)$$

$$K_{ij}^{13} = B^s \int_0^L \psi_{i,x} \psi_{j,x} dx \quad (12c)$$

$$K_{ij}^{22} = D \int_0^L \varphi_{i,xx} \varphi_{j,xx} dx \quad (12d)$$

$$K_{ij}^{23} = -D^s \int_0^L \varphi_{i,xx} \psi_{j,x} dx \quad (12e)$$

$$K_{ij}^{33} = H^s \int_0^L \psi_{i,x} \psi_{j,x} dx + A_s \int_0^L \psi_i \psi_j dx \quad (12f)$$

$$F_i = \int_0^L q \varphi_i dx \quad (12g)$$

The solution field derived from Eq. (11) enables to calculate the displacement and stresses of the LC beams.

2.4 Choice of shear function

It is noted from Eq. (1) that the accuracy of the HSDTs strictly depends on a suitable choice of the shear function $f(z)$. This topic has attracted a number of researches with different approaches. Table 1 introduces some representative shear functions for the HSDTs. Nguyen et al. [25] recently proposed a unified function and successfully applied for the analysis of composite plates. This approach will be used in the sequel to derive a shear function for the present HSDT.

Table 1. Shear functions for higher-order shear deformation theories

Reference	$f(z)$
Reddy [26]	$f(z) = z \left(1 - \frac{4z^2}{3h^2} \right)$
Touratier [27]	$f(z) = \frac{h}{\pi} \sin \left(\frac{\pi z}{h} \right)$
Soldatos [28]	$f(z) = h \sinh \left(\frac{z}{h} \right) - z \cosh \left(\frac{1}{2} \right)$

In order to derive a unified shear function for LC HSDT beams, it is supposed to approximate the shear function under the third-order polynomial as follows:

$$f(z) = d_1 z + d_3 z^3 \quad (13a)$$

$$g(z) = d_1 + 3d_3 z^2 \quad (13b)$$

where the relations of d_1, d_3 can be determined from the traction-free boundary conditions on the top and bottom faces of the beams, i.e. $g \left(z = \pm \frac{h}{2} \right) = 0$. The shear function $f(z)$ is finally obtained:

$$f(z) = d_1 z \left(1 - \frac{4z^2}{3h^2} \right) \quad (14)$$

with $d_1 = -\frac{3h^2}{4} d_3$. Moreover, by balancing the derivative of shear function given in Eq. (13b) with the other ones in Table 1 at $z=0$ leads to the different equivalent shear functions as reported in Table 2.

Table 2. Unified shear shape functions

Reference	d_1	d_3
Reddy [26]	1	$\frac{-4}{3h^2}$
Touratier [27]	1.25	$\frac{-5}{3h^2}$
Soldatos [28]	1	$\frac{-4}{3h^2}$

3. NUMERICAL RESULTS

Static behaviors of LC beam are considered in this section throughout numerical examples in which cross-ply LC beams with symmetrical $0^\circ/90^\circ/0^\circ$ fibers and asymmetrical $0^\circ/90^\circ$ ones are investigated. The orthotropic composite material properties are followed: $E_1 = 241.5 \times 10^6$, $E_1 / E_2 = 25$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $\nu_{12} = 0.25$. For simply purpose, the numerical results are normalised by following expressions:

$$\bar{w} = \frac{wbhE_2 h^2 10^2}{qL^4} \quad (15a)$$

$$\bar{\sigma}_x = \frac{bh^2}{qL^2} \sigma_x \left(\frac{L}{2}, \frac{h}{2} \right) \quad (15b)$$

$$\bar{\sigma}_{xz} = \frac{bh}{qL} \sigma_{xz} (0, 0) \quad (15c)$$

Table 3. Comparison of normalised deflection and stresses of $0^\circ/90^\circ/0^\circ$ LC beams with $L/h=5$

Reference	Variable	$f(z)$ reference	$f(z)$ equivalent
Reddy [26]	\bar{w}	2.4126	2.4126
	$\bar{\sigma}_{xx}$	1.0724	1.0724
	$\bar{\sigma}_{xz}$	0.4040	0.4040
Touratier [29]	\bar{w}	2.4420	2.4126
	$\bar{\sigma}_{xx}$	1.0970	1.0724
	$\bar{\sigma}_{xz}$	0.4218	0.4040
Soldatos [30]	\bar{w}	2.4097	2.4127
	$\bar{\sigma}_{xx}$	1.0702	1.0724
	$\bar{\sigma}_{xz}$	0.4023	0.4040

For verification purpose, Tables 3 and 4 present nondimensional deflections, axial and shear stresses of simply-supported $0^\circ/90^\circ/0^\circ$ LC beams with $L/h=5$ under uniformly distributed load. It is noted that the results reported in these tables are calculated with the equivalent shear functions associated with those of Reddy [26], Touratier [28] and Soldatos [30]. It can be seen that the results obtained from the present HSDT beam with equivalent $f(z)$ are in excellent agreements with those of previous works for both symmetric and asymmetric LC beams.

Table 4. Comparison of normalised deflection and stresses of $0^\circ/90^\circ$ LC beams with $L/h=5$

Refence	Value	$f(z)$ reference	$f(z)$ equivalent
Reddy [26]	\bar{w}	4.7771	4.7771
	$\bar{\sigma}_{xx}$	0.2363	0.2363
	$\bar{\sigma}_{xz}$	0.9133	0.9133
Touratier [29]	\bar{w}	4.7412	4.7771
	$\bar{\sigma}_{xx}$	0.2359	0.2363
	$\bar{\sigma}_{xz}$	0.9261	0.9133
Soldatos [30]	\bar{w}	4.7801	4.7770
	$\bar{\sigma}_{xx}$	0.2363	0.2363
	$\bar{\sigma}_{xz}$	0.9120	0.9133

Table 5. Comparison of normalised deflection of $[0^\circ/90^\circ/0^\circ]$ LC beams

BC	Reference	L/h			
		5	10	20	50
S-S	Aguiar [21]	2.426	1.105	0.762	0.665
	Khdeir [22]	2.412	1.096	-	0.665
	Murthy [23]	2.398	1.090	-	0.661
	Zenkour [24]	2.414	1.098	-	0.666
	Vo [4]	2.414	1.098	0.761	0.666
	Present1	2.412	1.096	0.759	0.664
	Present2	2.440	1.105	0.762	0.665
C-C	Present3	2.409	1.095	0.759	0.664
	Khdeir [22]	1.537	0.532	-	0.147
	Chakraborty [25]	1.629	0.504	-	0.144
	Present1	1.535	0.530	0.234	0.146
	Present2	1.528	0.534	0.237	0.147
	Present3	1.357	0.530	0.235	0.146

Moreover, in order to verify the accuracy of present the theory further in predicting the bending responses, Table 5 introduces the comparison of nondimensional deflection and stresses of the present model with three equivalent shear functions given in Table 2 and other earlier studies. The results are computed for $0^\circ/90^\circ/0^\circ$ LC beams with various values of span-to-thickness ratios $L/h=5$ and different boundary conditions. Good agreements between the beam models are again found. It again indicates the accuracy of the present model.

4. CONCLUSIONS

A unified third-order shear deformation beam theory is proposed in this paper. The theory is based on a higher-order variation of axial displacement through the beam thickness. The general third-order polynomial shear function is formulated from the traction-free boundary conditions and balance condition of the derivative of the present shear function with those from previous studies. Numerical results are obtained for both cross-ply symmetric and asymmetric laminated composite which is compared with earlier works, showed the accuracy and efficiency of the present model.

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