

APPLICATION OF ISOGEOMETRIC ANALYSIS TO FREE VIBRATION OF TRUSS STRUCTURES

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ABSTRACT

The Finite Element Method (FEM) is a standard tool to find natural vibration modes and frequencies of a structure, but the main problem is related with the inaccurate higher frequencies of a sample. Recently, there has appeared a new numerical method to solve partial differential equations, called Isogeometric Analysis (IGA), which uses NURBS (Non Uniform Rational B-Splines) as weighting functions to the solution of the problem and also as approach to the object geometry. In this paper, a study on the free vibration analysis based on isogeometric approach of truss structures is presented. Three refinement schemes such as h -, p - and k - refinement are used to obtain the accuracy of solution. Numerical results are then explored to show the effectiveness and accuracy of the present method by comparing its performance with finite element method (FEM), composite element method (CEM) and generalized finite element method (GFEM) reported in the literature. Based on the results, IGA is a promising tool, giving accurate results and high convergence rates.

Keywords: NURBS; IGA; Isogeometric analysis; truss; free vibration

TÓM TẮT

Phương pháp phần tử hữu hạn là công cụ chuẩn trong việc xác định dạng và tần số dao động của kết cấu, tuy nhiên kết quả thường có độ chính xác không cao. Gần đây phương pháp đẳng hình học được giới thiệu như một phương pháp số trong việc giải các phương trình vi phân. Nghiên cứu phân tích dao động tự do của kết cấu dựa trên phương pháp đẳng hình học được trình bày trong bài báo này. Để tăng độ chính xác, ba phương pháp làm mịn h , p và k được sử dụng trong phân tích. Độ chính xác và hiệu quả của phương pháp này sẽ được so sánh với phương pháp phần tử hữu hạn và phương pháp phần tử composite đã được nghiên cứu trước đó. Dựa trên kết quả nghiên cứu, phương pháp đẳng hình học là một công cụ đầy hứa hẹn, cho kết quả chính xác và tốc độ hội tụ cao.

Từ khóa: NURBS; IGA; phân tích đẳng hình học; dàn; dao động tự do.

1. INTRODUCTION

Nowadays, engineering structures are becoming taller, slender, light and cheaper. The dynamics effects are getting much more significant. Accurate and efficient numerical procedures must be developed in order to design safe structures.

Several vibration analysis methods can be found in the literature. Analytical solutions of the free vibration problems are known only for simple problems due to the difficulties of obtaining analytical solutions

for complex structures. In spite of these shortcomings, the analytical solution has been used recently to analyze the vibration of uniform and non-uniform bars [1,2]. In practice, the free vibration analysis is mainly obtained by the finite element method, which is widely applied in engineering design. In order to improve the accuracy of the results using the FEM, a mesh refinement must be done in order to improve the accuracy of the results for dynamic problems but the computational effort becomes much bigger and in many researches shown the accuracy for higher vibration modes is not improved. Many researchers have developed various numerical methods for improving the efficiency of the numerical solutions, for example, the composite element method presented by Zeng [3], or the GFEM proposed by Babuska [4,5].

In the recent years, the isogeometric analysis (IGA) is introduced by Hughes et al. [6]. This method allows one to integrate the computer aided geometric design (CAGD) representations directly into the element finite formulation. The isogeometric finite element formulation uses Non-uniform rational basis spline (NURBS) instead of the Lagrange interpolation in the FEM. The NURBS can provide higher continuity of derivatives in comparison with Lagrange interpolation functions. In addition, the order of the NURBS function can be easily elevated without changing the geometry or its parameterization [7,8].

In this paper, the author presents a formulation of the isogeometric analysis to free vibration analysis of truss structures. The IGA solution can be improved using three strategies: h-, p- and k-refinements. The efficiency and convergence of the method for vibration analysis truss structures are

checked. The frequencies obtained by the IGA are compared with those obtained by the FEM, CEM and GFEM.

The paper is structured as follows: Section 2 describes a brief of the isogeometric analysis. Isogeometric mode analysis of the truss structures is presented in section 3. Section 4 presents some applications of the proposed IGA. Finally, we close our paper with some concluding remarks.

2. THE ISOGEOMETRIC ANALYSIS

2.1 Knot vector

A knot vector in one dimension is a collection of non-decreasing set of coordinates or knots in the parameter space. The knot span is the interval between two knots. Knots divide a patch into elements. The knot vector can be represented as $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$, where $\xi_i \in \mathbb{R}$, i is the knot index, $i = 1, 2, \dots, n + p + 1$, p is the polynomial order and n is the number of the basis function used to construct the B-Spline curve. A knot vector is said to be open if its first and last knots are repeated $p+1$ times. It should be noted that open knot vectors are employed throughout this study.

2.2 Basis functions

The basis functions can be calculated using Cox-De Boor recursion formula with knot vector in hand. The basis functions can be defined recursively starting with $p = 0$ as:

$$N_{1,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For $p = 1, 2, 3, \dots$, they are defined by

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad (2)$$

The product of basis functions and the control point coordinate vector will give an

approximation for the curve, thus obtained using B-splines. The piecewise polynomial B-spline curve $C(\xi)$ is given by

$$C(\xi) = \sum_{i=1}^n N_{i,p}(\xi) \mathbf{P}_i \quad (3)$$

where \mathbf{P}_i refers to the control point coordinates. Fig. 1 shows an example of the quadratic B-spline basis functions for the open and non-uniform knot vectors.

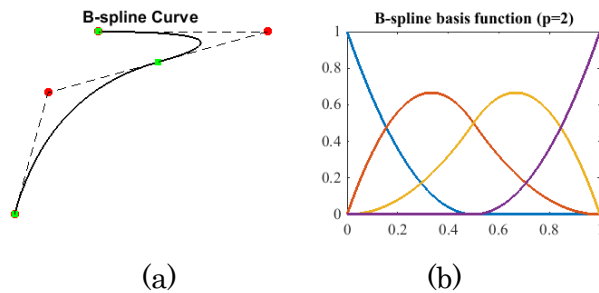


Fig. 1 Quadratic B-splines curves:

a) Quadratic B-spline curve; b) Basis functions.

NURBS basis is defined by associating the B-spline basis functions with a positive term called weight, ω_i such that

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)\omega_i}{\sum_{i=1}^n N_{i,p}(\xi)\omega_i} \quad (4)$$

The NURBS curve components are the linear combination of the NURBS basis functions weighted by the components of control points

$$C(\xi) = \sum_{i=1}^n R_{i,p}(\xi) \mathbf{P}_i \quad (5)$$

2.3 Refinement

In order to accurately predict the physical behaviour and thereby increase the accuracy of the solution, the mesh might have to be refined. The different methods of refinement are knot insertion, degree elevation, and degree and continuity elevation.

The knot insertion method is called h -refinement. The new knot vector is generated by adding more knots to the existing knot vector in this method. The B-spline curve generated from the new knot vector has more number of control points and hence more number of elements. Fig. 2 shows an example h refinement.

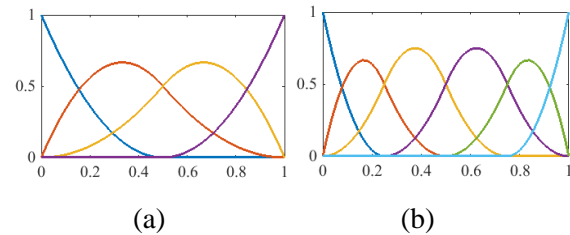


Fig. 2 Example of Isogeometric h refinement:

a) original knot vector $\Xi = \{0, 0, 0, 0.5, 1, 1, 1\}$

b) new knot vector $\Xi = \{0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1\}$.

The second way for refining the basis and enriching the knot vector is to increase the order of the basis function, which is called p -refinement. The basis has $p - m_i$ continuous derivatives across element boundaries, where m_i is the multiplicity of a knot. Unlike knot insertion, order elevation affects the curve globally. Fig. 3 shows an example p refinement.

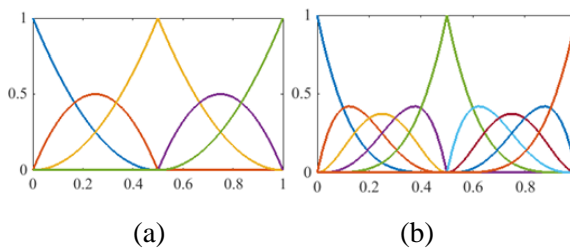


Fig. 3 Example of Isogeometric p refinement:

a) original knot vector $\Xi = \{0, 0, 0, 0.5, 0.5, 1, 1, 1\}$

b) new knot vector

$\Xi = \{0, 0, 0, 0, 0, 0.5, 0.5, 0.5, 1, 1, 1, 1\}$

The final method is k refinement which both mesh refinement and degree elevation are carried out. Fig. 4 illustrates an example of k refinement.

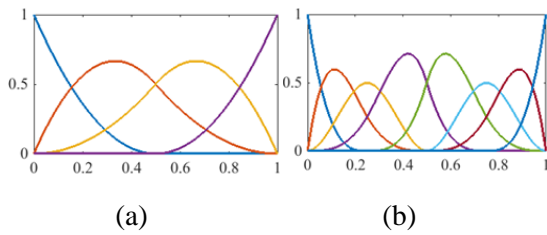


Fig. 4 Example of Isogeometric k refinement:
 a) original knot vector $\Xi = \{0, 0, 0, 0.5, 0.5, 1, 1, 1\}$ b)
 new knot vector
 $\Xi = \{0, 0, 0, 0, 0.25, 0.5, 0.5, 0.75, 1, 1, 1, 1\}$

3. ISOGEOMETRIC FORMULATION FOR FREE VIBRATION OF TRUSS STRUCTURES

3.1 Displacement of longitudinal bar element

Consider a bar shown in Fig. 5 where the local x -axis is taken in the axial direction of the element with its origin at local node 1. We can construct the displacement function $u(x)$ as:

$$u(x) = R_{1,p}q_1 + \sum_{i=2}^{n-1} R_{i,p}c_{i-1} + R_{n,p}q_2 \quad (6)$$

where $R_{1,p} \dots R_{n,p}$ the p^{th} -degree NURBS basis function, q_1 , q_2 and c_i are the first, last and interior nodal displacement, n is number of control points or NURBS functions.

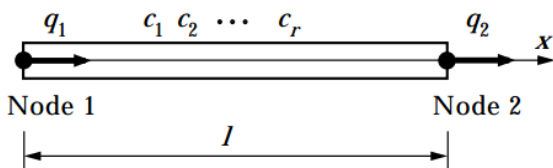


Fig. 5 Constructing of IGA for longitudinal bar.

It is easy to derive the stiffness matrix \mathbf{k}_e and mass matrix \mathbf{m}_e of the IGA by the general procedure of the conventional FEM.

$$\mathbf{k}_e = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV \quad (7)$$

$$\mathbf{m}_e = \int_V \rho \mathbf{R}^T \mathbf{R} dV \quad (8)$$

where \mathbf{k}_e is the elastic matrix, the superscript e denotes each element and \mathbf{B} is the strain-displacement matrix.

3.2 Global stiffness matrix, mass matrix and coordinate transformation

In order to find the stiffness matrix and the mass matrix of the bar element of the IGA in the global coordinate system, we need to search a transformation matrix. Let the transformation matrix \mathbf{T}^e exist between the local and the global coordinate systems such that

$$\mathbf{u}_e = \mathbf{T}^e \mathbf{u}_g \quad (9)$$

where $\mathbf{u}_e = [q_1 \ c_1 \ c_2 \ \dots \ q_2]^T$ and

$\mathbf{u}_g = [u_i \ v_i \ u_j \ v_j \ c_1 \ c_2 \ \dots]^T$ are the coordinate of element e in the local system and global system, respectively. The stiffness matrix \mathbf{K}_e and mass matrix \mathbf{M}_e of the element corresponding to the global coordinate system are given as

$$\mathbf{K}_e = \mathbf{T}^{eT} \mathbf{k}_e \mathbf{T}^e \quad (10)$$

$$\mathbf{M}_e = \mathbf{T}^{eT} \mathbf{m}_e \mathbf{T}^e \quad (11)$$

For planar case, the local nodes 1 and 2 of the bar element correspond to nodes i and j respectively of the global system as shown in Fig. 6

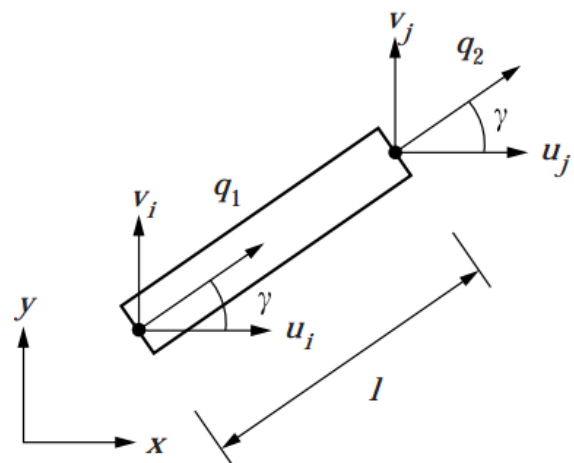


Fig. 6 Coordinate transformation in planar case.

The local displacements q_1 and q_2 can be resolved into components u_i, v_i and u_j, v_j parallel to the global x, y axes respectively. The two sets of displacements are related as

$$\begin{aligned} q_1 &= u_i \cos \gamma + u_j \sin \gamma \\ q_2 &= v_j \cos \gamma + v_i \sin \gamma \end{aligned} \quad (12)$$

where γ is the angle between the line ij and the direction ox . According to the transformation relation Eq (12), the displacement field the element can be written as

$$u(x) = \mathbf{R}^e(x) \mathbf{T}^e \mathbf{u}^e \quad (13)$$

where $\mathbf{R}^e(x)$ is NURBS shape function matrix, \mathbf{T}^e is the transformation matrix.

$$\mathbf{T}^e = \begin{bmatrix} l_{ij} & m_{ij} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & l_{ij} & m_{ij} & 0 & 0 & \dots & 0 \end{bmatrix} \quad (14)$$

The direction cosines l_{ij}, m_{ij} can be computed in terms of the global coordinates of nodes i and j as

$$l_{ij} = \cos \gamma = \frac{x_j - x_i}{l}, \quad m_{ij} = \sin \gamma = \frac{y_j - y_i}{l} \quad (15)$$

where (x_i, y_i) and (x_j, y_j) are the global coordinates of nodes i and j respectively, and l is the length of the element e given by

$$l = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \quad (16)$$

With the help of the expression of transformation matrix Eq (14), we can compute the stiffness and mass matrices in the global coordinate system with Eq (10) and Eq (11).

4. NUMERICAL EXAMPLES

4.1 Seven bar truss

Let us consider a more complicated truss structure composed of seven bars shown in Fig.7 and find the natural frequencies and modes of free vibration. This problem was proposed by Zeng [9] in order to check Composite Element Method. All bars in the truss have cross section area $A = 0.001 \text{ m}^2$, mass density $\rho = 8000 \text{ kg/m}^3$ and elasticity modulus $E = 2.1 \times 10^{11} \text{ N/m}^2$. The frequencies obtained by each analysis are presented in Table 1.

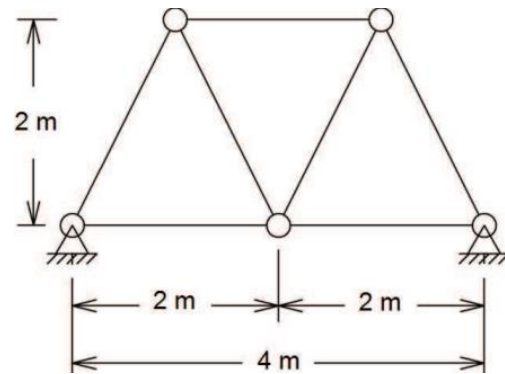


Fig. 7 Seven bar truss.

The results of Table 1 show us that the IGA can obtain higher accuracy natural frequencies than the conventional FEM, CEM and GFEM in previous works. The natural mode configurations shown in Fig. 8-Fig. 11 also reflect the difference in the results between the FEM and the IGA.

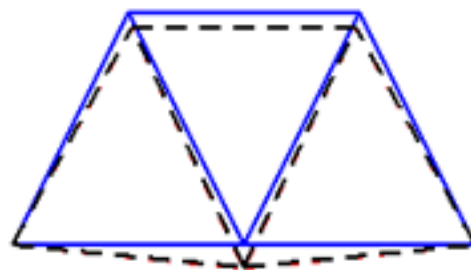


Fig. 8 The first-order natural mode of 7 bar truss: red color by FEM; black color by IGA.

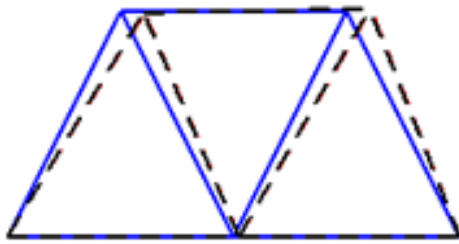


Fig. 9 The second-order natural mode of 7 bar truss: red color by FEM; black color by IGA.

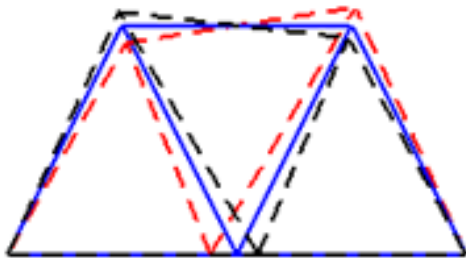


Fig. 10 The fourth-order natural mode of 7 bar truss: red color by FEM; black color by IGA.

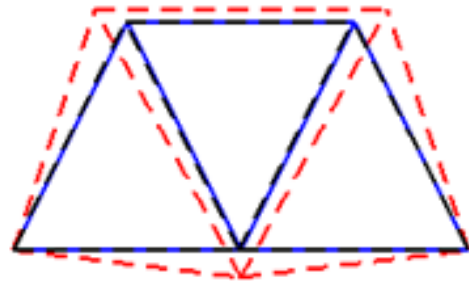


Fig. 11 The sixth-order natural mode of 7 bar truss: red color by FEM; black color by IGA.

The results show that the c-version of CEM, the adaptive GFEM and the IGA converge to the same frequencies.

4.2 Fifteen bar truss

The free axial vibration of a truss formed by 15 straight bars is analyzed to illustrate the application of the IGA. This problem was proposed by Zeng [9] in order to check Composite Element Method. The geometry of the truss is presented in Fig. 12. All bars in the truss have cross section area $A=0.001 \text{ m}^2$,

mass density $\rho = 8000 \text{ kg/m}^3$ and elasticity modulus $E = 2.1 \times 10^{11} \text{ N/m}^2$.

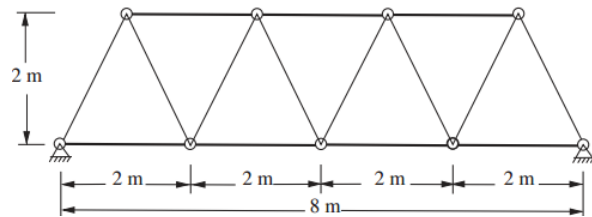


Fig. 12 Fifteen bar truss.

We take h -, k - and p - refinement for each element of truss, then compare the calculated natural frequencies with the results of FEM, CEM and GFEM, which are shown in Table 2.



Fig. 13 The first-order natural mode of 15 bar truss: red color by FEM; black color by IGA.

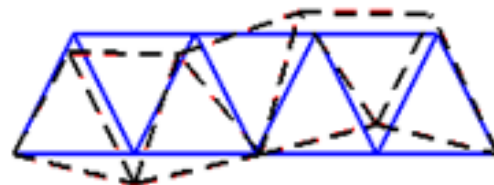


Fig. 14 The third-order natural mode of 15 bar truss: red color by FEM; black color by IGA.



Fig. 15 The fourth-order natural mode of 15 bar truss: red color by FEM; black color by IGA.

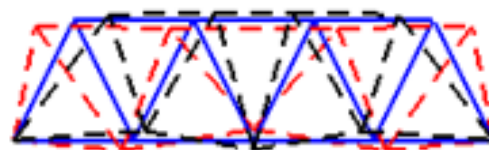


Fig. 16 The sixth-order natural mode of 15 bar truss: red color by FEM; black color by IGA.

The natural mode configurations shows in Fig.13 - Fig.16. The difference in the results between the FEM and the IGA for lower-order modes, the error between them is not large, but for higher-order modes, the error is obvious.

5. CONCLUSIONS

The main contribution of this study consists of proposing the isogeometric analysis for vibration analysis of truss

structures. The results show that the IGA is more accurate than the CEM, the adaptive GFEM results. Results for the case of the IGA with k-refined are the same as that with p-FEM method. Three methods in previous works employed a larger number of degrees of freedom. Thus the IGA has shown to be efficient in the analysis of vibration of truss structures so that it can be applied to complex problems in practice.

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TABLES

Table 1: Results to free vibration of seven bar truss

Mode (<i>i</i>) - ω_i (rad/s)	01	02	03	04	05	06
FEM (7e-linear), ndof =6	1683.52	1776.27	3341.37	5147.35	5678.18	8315.40
FEM (7e-quadtic), ndof =13	1648.06	1741.21	3117.08	4597.86	4869.00	7430.30
CEM (7e-2c) , ndof =20[9,10]	1648.26	1741.32	3113.83	4567.69	4829.70	7379.96
CEM (7e-5c) , ndof =41[9,10]	1647.81	1740.86	3111.52	4562.56	4824.12	7379.51
GFEM (7e) , ndof =34[9,10]	1647.785	1740.840	3111.326	4561.819	4823.253	7379.482
Adaptive GFEM(7e), ndof =74 [10]	1647,784	1740.839	3111.322	4561.817	4823.248	7379.482
IGA(7e), ndof =6	1683.52	1776.27	3341.37	5147.35	5678.18	8315.40
IGA(14e), ndof =13, <i>h-refined</i>	1656.66	1749.74	3168.41	4726.01	5045.20	8134.14
IGA (7e), ndof =13, <i>k-refined</i>	1648.06	1741.21	3117.08	4597.86	4869.00	7430.30
IGA (7e), ndof =20, <i>k-refined</i>	1647.785	1740.840	3111.403	4562.601	4825.087	7429.357
IGA (7e), ndof =27, <i>p-refined</i>	1647.80	1740.86	3111.71	4564.26	4826.76	7407.98

7e = 7 elements

7e-2c = 7 elements and 2 interior degrees of freedom per element

ndof = number of degrees of freedom after introduction of boundary conditions.

Table 2: Results to free vibration of fifteen bar truss

Mode (<i>i</i>)	01	02	03	04	05	06
	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_4 (rad/s)	ω_5 (rad/s)	ω_6 (rad/s)
FEM (15e-linear), ndof =14	682.27	1149.29	1612.35	2519.86	2715.75	2968.22
FEM (15e-quadtic), ndof =29	679.78	1139.23	1581.99	2410.63	2602.86	2816.73
CEM (15e-2c) , ndof =44 [9,10]	679.82	1139.34	1582.18	2410.25	2601.85	2815.44
CEM (15e-5c) , ndof =74 [9,10]	679.79	1139.22	1581.83	2409.13	2600.64	2813.92
Adaptive GFEM, ndof =74 [10]	679.786	1139.200	1581.771	2408.911	2600.405	2813.617
IGA(15e), ndof =14	682.27	1149.29	1612.35	2519.86	2715.75	2968.22
IGA(15e), ndof =29, <i>h-refined</i>	680.406	1141.721	1589.386	2436.364	2629.545	2852.098
IGA (15e), ndof =29, <i>k-refined</i>	679.789	1139.237	1581.995	2410.634	2602.863	2816.731
IGA (15e), ndof =44, <i>k-refined</i>	679.786	1139.200	1581.772	2408.926	2600.420	2813.648
IGA (15e), ndof =29, <i>p-refined</i>	679.789	1139.237	1581.995	2410.634	2602.863	2816.731

15e = 15 elements

15e-2c = 15 elements and 2 interior degrees of freedom per element

ndof = number of degrees of freedom after introduction of boundary conditions.