USING TRANSFORMATION MATRIX TO ANALYZE PLANAR MECHANISM KINEMATICS

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ABSTRACT

Planar mechanism kinematic analysis can be performed either analytically or graphically. Graphical kinematic analysis is considered as a simple, intuitive approach but less accurate because values of kinematic quantities are measured from graphical vector diagrams. Analytical kinematics is a more advanced method thanks to using precise mathematical operations and easy to automate. Currently, there are many analytical kinematic methods introduced in somedocuments in universities but most of them are difficult to use because of complicated application.Finding an appropriate approach to solve more easily kinematic problems for planar mechanisms is a necessary work. By using the transformation matrix and using basic operations for matrix (such as addition, scalar multiplication, and derivative) the position, velocity, and acceleration equationswill be established for a planar mechanism. From these formulas, analytical kinematics can be applied for typical planar mechanisms. The first advantage of this method is able to calculate position, velocity, and acceleration of links and joints accurately at any position of input link. The second is to develop computation process automatically thanks to support from computer program, such as MatLab, Excel. Some chosen examples shown out here aim todemonstrate the transformation matrix application to solve analytical kinematic problems for different planar mechanisms.

Keywords: *Transformation matrix, Analytical kinematics, Planar mechanism*

TÓM TẮT

Phân tích động học cơ cấu phẳng có thể được thực hiện bằng phương pháp giải tích hoặc phương pháp hình học. Phân tích động học bằng giải tích được xem như là phương pháp đơn giản, trực quan nhưng thiếu chính xác do các đại lượng động học có giá trị được đo từ các họa đồ vec tơ. Động học giải tích là một phương pháp tiên tiến hơn nhờ dùng các phép toán chính xác và dễ thực hiện tự động. Hiện nay có nhiều phương pháp động học giải tích được giới thiệu trong các tài liệu ở các trường đại học nhưng hầu hết chúng rất khó dùng vì tính phức tạp khi áp dụng. Tìm ra một phương pháp thích hợp để giải các bài toán động học cơ cấu phẳng là một công việc cần thiết. Bằng cách sử dụng ma trận chuyển vị và dùng các phép toán ma trận như cộng ma trận, nhân ma trận, đạo hàm ma trận thì các phương trình vị trí, vận tốc, gia tốc sẽ được thiết lập cho cơ cấu phẳng. Từ các phương trình này, động học giải tích có thể áp dụng cho các cơ cấu phẳng điển hình. Ưu điểm trước hết của phương pháp này là có thể tính toán vị trí, vận tốc và gia tốc của các khâu và các khớp trong cơ cấu chính xác theo bất kỳ vị trí nào của khâu dẫn. Ưu điểm thứ hai của phương pháp là phát triển quá trình tính toán theo hướng tự động nhờ sự hỗ trợ của các phần mềm máy tính như MatLab, Excel. Một vài ví dụ được chọn ra ở đây nhằm mục đích minh họa khả năng áp dụng ma trận chuyển vị để giải bài toán động học cho các cơ cấu phẳng khác nhau theo hướng giải tích.

Từ khóa: *Ma trận chuyển vị, Động học giải tích, Cơ cấu phẳng*

1. INTRODUCTIONS

In planar mechanism, kinematic analysis can be performed either analytically or graphically [1]. Graphical kinematic analysis is considered as a simple, intuitive method but less accurate and very difficult to use by computer-aided method. Analytical kinematics is based on the matrix called transformation matrix which is established by trigonometric functions in a term of rotation angle of links. Then, for a given value of the position (or orientation) of the input link, the algebraic equations are solved for the position and/or orientation of the remaining links. The first and second time derivative of the algebraic position equations will provide the velocity and acceleration equations for the mechanism. For given value of the velocity and acceleration of the input link, these equations are solved to find the velocity and acceleration of the other links in the system [1].

Analytical kinematics is a systematic process that is most suitable for developing into a computer program [1], such as MatLab, Excel. The paper contains two parts: the first is to denote theoretical basis, the second is to demonstrate examples.

2. THEORETICAL BASIS

A mechanism (or linkage) is considered as a collection of the links that are interconnected by kinematic joints forming a single or multiple degree-of-freedom chain. One link is designated the frame because it served as the frame of reference for the motion of all other links [2]. Links are the individual parts which are considered rigid bodies. Theoretically, a true rigid body does not change shape during motion. A joint is a movable connection between and allows relative motion between the links. The two primary joints are revolute and sliding joint. Linkage can be either open or closed chains (Fig. 1,2) [2].

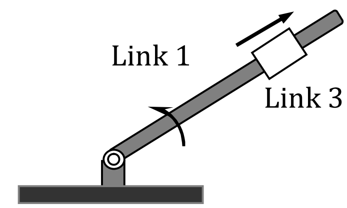

Fig. 1: *The open mechanism*

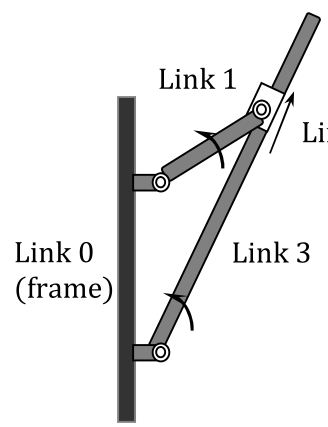

Fig. 2: *The closed mechanism*

In order to establish a general formula for kinematic analysis, it had better to consider kinematic of one ith link in Fig. 3that is moving in a plane. The point *A* fixes on the link while the point *B* moves on a straight line. We design two coordinate systems in which one coordinate system named $O_0X_{01}X_{02}$ is fixed to the frame (the fixed coordinate system) and the other with name $O_iX_{i1}X_{i2}$ is fixed to the ith link (the moving coordinate system). Here, *A* denotes the joint, O_i coincides with A, and O_iX_{i1} coincides with AB. The kinematic input quantities are known as [3]:

$$\begin{cases} x_{A1} = x_{A1}(t) \\ x_{A2} = x_{A2}(t) \\ \theta_i = \theta_i(t) \\ s_i = s_i(t) \end{cases}$$

Where $\boldsymbol{O_0A} = \boldsymbol{r_A} = (x_{A1}, x_{A2})$ is vector specified position of point A; (x_{A1}, x_{A2}) is coordinates of point A. The quantity θ_i denotes revolution angle of the link. The quantity s_i denotes the first coordinate of point B and t denotes time. *Here, bold characters express vector quantities.*

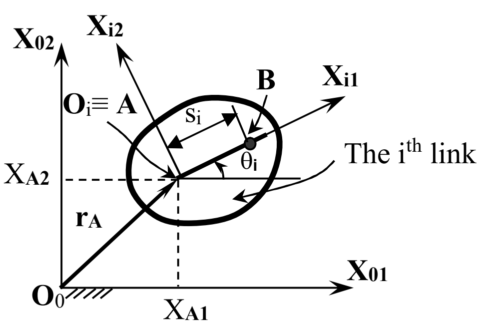

Fig. 3: *General kinematic model of planar mechanism*

The position of point B can be written in a term of $x_{A1}, x_{A2}, \theta_i, s_i$ as following:

$$\begin{cases} x_{B1} = x_{A1} + s_i.\cos\theta_i \\ x_{B2} = x_{A2} + s_i.\sin\theta_i \end{cases}$$

Or

$$\begin{Bmatrix} x_{B1} \\ x_{B2} \end{Bmatrix} = \begin{Bmatrix} x_{A1} \\ x_{A2} \end{Bmatrix} + \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} . \begin{Bmatrix} s_i \\ 0 \end{Bmatrix}$$

It can rewrite in a vector form [4]:

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{T}_i.\boldsymbol{u}_i \quad (1)$$

Where $\boldsymbol{r_B} = \begin{Bmatrix} x_{B1} \\ x_{B2} \end{Bmatrix}$ is the vector specified position of point B in the fixed coordinate system.

$\boldsymbol{r_A} = \begin{Bmatrix} x_{A1} \\ x_{A2} \end{Bmatrix}$ is the vector specified position of point A in the fixed coordinate system.

$\boldsymbol{u_i} = \boldsymbol{AB} = \begin{Bmatrix} s_i \\ 0 \end{Bmatrix}$ is the vector specified position of point B in the moving coordinate system.

$\boldsymbol{T_i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$ is the transformation matrix of the i^th link compare to the fixed coordinate system [4].

Then, the velocity of point B is established from the first time derivative of the position equation (1).

$$\dot{\boldsymbol{r}}_B = \dot{\boldsymbol{r}}_A + \dot{\boldsymbol{T}}_i.\boldsymbol{u}_i + \boldsymbol{T}_i.\dot{\boldsymbol{u}}_i$$

Where $\dot{\boldsymbol{r}}_B = \begin{Bmatrix} \dot{x}_{B1} \\ \dot{x}_{B2} \end{Bmatrix}$ is the velocity of point B in the fixed coordinate system (absolute velocity).

$\dot{\boldsymbol{r}}_A = \begin{Bmatrix} \dot{x}_{A1} \\ \dot{x}_{A2} \end{Bmatrix}$ is the velocity of point A in the fixed coordinate system (absolute velocity).

$\dot{\boldsymbol{u}}_i = \begin{Bmatrix} \dot{s}_i \\ 0 \end{Bmatrix}$ is the velocity of point B in the moving coordinate system (relative velocity).

$$\dot{\boldsymbol{T}}_i = \begin{bmatrix} -\sin\theta_i & -\cos\theta_i \\ \cos\theta_i & -\sin\theta_i \end{bmatrix} . \dot{\theta}_i$$
$$= \omega_i . \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$$
$$= \omega_i . \boldsymbol{I}.\boldsymbol{T}_i$$

Here, $\omega_i = \dot{\theta}_i$ is the angular velocity of the i^th link, $\boldsymbol{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the constant matrix.

Deduce

$$\dot{\boldsymbol{r}}_B = \dot{\boldsymbol{r}}_A + \omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\boldsymbol{u}_i + \boldsymbol{T}_i.\dot{\boldsymbol{u}}_i$$

Or in the other form the velocity equation can be rewritten:

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\boldsymbol{u}_i + \boldsymbol{T}_i.\dot{\boldsymbol{u}}_i \qquad (2)$$

The second and the third components on the right side of the equation (2) characterize the changes of direction and magnitude of the position vector $\boldsymbol{u}$ in the fixed coordinate system.

Continuously, the acceleration of point B is received from the first time derivative of the velocity equation (2).

$$\dot{\boldsymbol{v}}_B = \dot{\boldsymbol{v}}_A + \dot{\omega}_i.\boldsymbol{I}.\boldsymbol{T}_i.\boldsymbol{u}_i + \omega_i.\boldsymbol{I}.\dot{\boldsymbol{T}}_i.\boldsymbol{u}_i + \omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\dot{\boldsymbol{u}}_i + \dot{\boldsymbol{T}}_i.\dot{\boldsymbol{u}}_i + \boldsymbol{T}_i.\ddot{\boldsymbol{u}}_i$$

Replace $\dot{\boldsymbol{T}}_i = \omega_i.\boldsymbol{I}.\boldsymbol{T}_i$ in the above equation, we get:

$$\dot{\boldsymbol{v}}_B = \dot{\boldsymbol{v}}_A + \dot{\omega}_i.\boldsymbol{I}.\boldsymbol{T}_i.\boldsymbol{u}_i + \omega_i.\boldsymbol{I}.\omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\boldsymbol{u}_i + \omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\dot{\boldsymbol{u}}_i + \omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\dot{\boldsymbol{u}}_i + \boldsymbol{T}_i.\ddot{\boldsymbol{u}}_i$$

Or $\dot{\boldsymbol{v}}_B = \dot{\boldsymbol{v}}_A + \dot{\omega}_i.\boldsymbol{I}.\boldsymbol{T}_i.\boldsymbol{u}_i + \omega_i^2.\boldsymbol{I}^2.\boldsymbol{T}_i.\boldsymbol{u}_i + 2\omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\dot{\boldsymbol{u}}_i + \boldsymbol{T}_i.\ddot{\boldsymbol{u}}_i$

Where $\boldsymbol{I}^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -\boldsymbol{E}$; here, $\boldsymbol{E}$ is the unit matrix.

The quantity $\dot{\omega}_i = \varepsilon_i$ is called angular acceleration of the i^th^ link, and the quantity $\ddot{\boldsymbol{u}}_i$ is relative acceleration of point B in the moving coordinate system.

Deduce

$$\dot{\boldsymbol{v}}_B = \dot{\boldsymbol{v}}_A + \varepsilon_i.\boldsymbol{I}.\boldsymbol{T}_i.\boldsymbol{u}_i - \omega_i^2.\boldsymbol{E}.\boldsymbol{T}_i.\boldsymbol{u}_i + 2\omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\dot{\boldsymbol{u}}_i + \boldsymbol{T}_i.\ddot{\boldsymbol{u}}_i$$

Or

$$\boldsymbol{a}_B = \boldsymbol{a}_A + (\varepsilon_i.\boldsymbol{I}.-\omega_i^2.\boldsymbol{E}).\boldsymbol{T}_i.\boldsymbol{u}_i + 2\omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\dot{\boldsymbol{u}}_i + \boldsymbol{T}_i.\ddot{\boldsymbol{u}}_i \qquad (3)$$

Where $\boldsymbol{a}_A, \boldsymbol{a}_B$ are the absolute acceleration of point A and B. The third component on a right side of the equation (3) is known as *Coriolic* acceleration [3].

In a special case, if point B is fixed to the i^th^ link, it means$s_i = const$ leading

$\dot{\boldsymbol{u}}_i = \ddot{\boldsymbol{u}}_i = 0$. Equations (2), (3) can be rewritten as

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \omega_i.\boldsymbol{I}.\boldsymbol{T}_i.\boldsymbol{u}_i \qquad (4)$$

$$\boldsymbol{a}_B = \boldsymbol{a}_A + (\varepsilon_i.\boldsymbol{I}.-\omega_i^2.\boldsymbol{E}).\boldsymbol{T}_i.\boldsymbol{u}_i \qquad (5)$$

3. EXAMPLES

Example 1:

Given: Link 0 = Frame; Link 1 = Rotation rod OB moving with $\theta = \theta(t)$; Link 2 = Slider A moving with $s = s(t)$

Determine: position, absolute velocity, and absolute acceleration of the slider A. (Fig. 4)

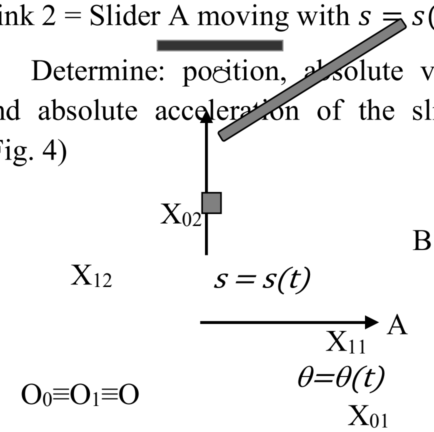

***Fig. 4**: Mechanism of slider on rotation rod*

Solution:

Chain of link: Frame + (revolute joint O) + Rotation rod OB + (sliding joint) + Slider A

Using equation (1): $\boldsymbol{r}_A = \boldsymbol{r}_O + \boldsymbol{T}_1.\boldsymbol{u}_1$

Where: $\boldsymbol{r}_O = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$; $\boldsymbol{T}_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$; $\boldsymbol{u}_1 = \begin{Bmatrix} s \\ 0 \end{Bmatrix}$

Leading position of slider A :

$$\boldsymbol{r}_A = \begin{Bmatrix} x_{A1} \\ x_{A2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.\begin{Bmatrix} s \\ 0 \end{Bmatrix} = \begin{Bmatrix} s.\cos\theta \\ s.\sin\theta \end{Bmatrix}$$

Using equation (2):

$$\boldsymbol{v}_A = \boldsymbol{v}_O + \omega_1.\boldsymbol{I}.\boldsymbol{T}_1.\boldsymbol{u}_1 + \boldsymbol{T}_1.\dot{\boldsymbol{u}}_1$$

Where: $\boldsymbol{v}_O = \begin{Bmatrix}0\\0\end{Bmatrix}$; $\boldsymbol{I} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$; $\dot{\boldsymbol{u}}_1 = \begin{Bmatrix}\dot{s}\\0\end{Bmatrix}$; $\omega_1 = \dot{\theta}$

Then, absolute velocity of slider A is determined

$$\boldsymbol{v}_A = \begin{Bmatrix}0\\0\end{Bmatrix} + \dot{\theta}.\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}.\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}.\begin{Bmatrix}s\\0\end{Bmatrix} + \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}.\begin{Bmatrix}\dot{s}\\0\end{Bmatrix}$$

$$\boldsymbol{v}_A = \begin{Bmatrix}v_{A1}\\v_{A2}\end{Bmatrix} = \begin{Bmatrix}-\dot{\theta}.s.\sin\theta + \dot{s}.\cos\theta\\\theta.s.\cos\theta + \dot{s}.\sin\theta\end{Bmatrix}$$

Using equation (3):

$$\boldsymbol{a}_A = \boldsymbol{a}_O + (\varepsilon_1.\boldsymbol{I}.-\omega_1^2.\boldsymbol{E}).\boldsymbol{T}_1.\boldsymbol{u}_1 + 2\omega_1.\boldsymbol{I}.\boldsymbol{T}_1.\dot{\boldsymbol{u}}_1 + \boldsymbol{T}_1.\ddot{\boldsymbol{u}}_1$$

Where: $\boldsymbol{a}_o = \begin{Bmatrix}0\\0\end{Bmatrix}$; $\boldsymbol{E} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$; $\ddot{\boldsymbol{u}}_1 = \begin{Bmatrix}\ddot{s}\\0\end{Bmatrix}$; $\varepsilon_1 = \ddot{\theta}$

In similar way, we get the final result.

$$\boldsymbol{a}_A = \begin{Bmatrix}a_{A1}\\a_{A2}\end{Bmatrix} = \begin{Bmatrix}-\dot{\theta}^2.s.\cos\theta - \ddot{\theta}.s.\sin\theta - 2\dot{\theta}.\dot{s}.\sin\theta + \dot{s}.\cos\theta\\-\dot{\theta}^2.s.\sin\theta + \ddot{\theta}.s.\cos\theta + 2\dot{\theta}.\dot{s}.\cos\theta + \dot{s}.\sin\theta\end{Bmatrix}$$

Example 2: (Culit mechanism) [2, 3]

Given: Link 0 = Frame; Link 1 = Crank O_1Arotating with$\omega = const$ and$\theta = \omega.t$; Link 2 = Slider A; Link 3 = rod O_3B; Length of O_1A = l ; Distance of $O_1O_2 = d$

Determine: position, sliding velocity, and sliding acceleration of the slider A on the rod O_3B; position, angular velocity, and angular acceleration of the rod O_3B. (Fig. 5)

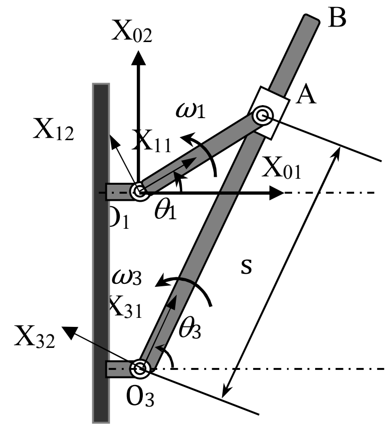

Fig. 5: *Culit Mechanism*

Solution:

This mechanism is a closed chain that consists of two open link chains.

The first chain of links: Frame + (revolute joint O_1) + Crank O_1A + (revolute joint A)

The second chain of links:

Frame + (revolute joint O_3) + Rotation rod O_3B + (sliding joint) + Slider A + (revolute joint A)

Two chains are closed at joint A.

Position of revolute joint A is determined as following.

With the first chain of linkage, applying equation (1) for this case:

$$\boldsymbol{r}_A = \boldsymbol{r}_{O1} + \boldsymbol{T}_1.\boldsymbol{u}_1$$

Where:

$\boldsymbol{r}_{O1} = \begin{Bmatrix}0\\0\end{Bmatrix}$; $\boldsymbol{T}_1 = \begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\\sin\theta_1 & \cos\theta_1\end{bmatrix}$; $\boldsymbol{u}_1 = \begin{Bmatrix}l\\0\end{Bmatrix}$; $\theta_1 = \theta = \omega.t$

With the second chain of linkage, in a similarity: $\boldsymbol{r}_A = \boldsymbol{r}_{O3} + \boldsymbol{T}_3.\boldsymbol{u}_3$

Where:

$\boldsymbol{r}_{O3} = \begin{Bmatrix} 0 \\ -d \end{Bmatrix}$; $\boldsymbol{T}_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{bmatrix}$; $\boldsymbol{u}_3 = \begin{Bmatrix} s \\ 0 \end{Bmatrix}$

Two open linkages must have a same position of point A. It means

$$\boldsymbol{r}_{O1} + \boldsymbol{T}_1.\boldsymbol{u}_1 = \boldsymbol{r}_{O3} + \boldsymbol{T}_3.\boldsymbol{u}_3$$

Replacement and combination of above equations, we get equation systems:

$$\begin{cases} s.\cos\theta_3 = l.\cos\theta_1 \\ -d + s.\sin\theta_3 = l.\sin\theta_1 \end{cases}$$

Solving two algebraic equations to get two unknown quantities be functions of given quantities as d, l, $\theta_1 = \theta = \omega.t$

Similarly, the velocity of revolute joint A is determined as following.

With the first chain of linkage, applying equation (4) for this case:

$$\boldsymbol{v}_A = \boldsymbol{v}_{O1} + \omega_1.\boldsymbol{I}.\boldsymbol{T}_1.\boldsymbol{u}_1$$

With the second chain of linkage, using equation (2):

$$\boldsymbol{v}_A = \boldsymbol{v}_{O3} + \omega_3.\boldsymbol{I}.\boldsymbol{T}_3.\boldsymbol{u}_3 + \boldsymbol{T}_3.\dot{\boldsymbol{u}}_3$$

Two open linkages must have a same velocity of point A. It means

$$\boldsymbol{v}_{O1} + \omega_1.\boldsymbol{I}.\boldsymbol{T}_1.\boldsymbol{u}_1 = \boldsymbol{v}_{O3} + \omega_3.\boldsymbol{I}.\boldsymbol{T}_3.\boldsymbol{u}_3 + \boldsymbol{T}_3.\dot{\boldsymbol{u}}_3$$

Where: $\boldsymbol{v}_{O1} = \boldsymbol{v}_{O3} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$; $\omega_1 = \omega$; $\boldsymbol{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; $\dot{\boldsymbol{u}}_3 = \begin{Bmatrix} \dot{s} \\ 0 \end{Bmatrix}$

Finally, the velocity equation system can be write as

$$\begin{cases} -\omega_3.s.\sin\theta_3 + \dot{s}.\cos\theta_3 = -\omega.l.\sin\theta \\ \omega_3.s.\cos\theta_3 + \dot{s}.\sin\theta_3 = \omega.l.\cos\theta \end{cases}$$

Here, we get two algebraic equations with two unknowns $\dot{s}$ and ω_3. These unknowns are solved in a term of θ, ω, l.

In the same as way, the acceleration of revolute joint A is also calculated.

With the first chain of linkage, applying equation (5):

$$\boldsymbol{a}_A = \boldsymbol{a}_{O1} + (\varepsilon_1.\boldsymbol{I}.-\omega_1^2.\boldsymbol{E}).\boldsymbol{T}_1.\boldsymbol{u}_1$$

With the second chain of linkage, using equation (3):

$$\boldsymbol{a}_A = \boldsymbol{a}_{O3} + (\varepsilon_3.\boldsymbol{I}.-\omega_3^2.\boldsymbol{E}).\boldsymbol{T}_3.\boldsymbol{u}_3 + 2\omega_3.\boldsymbol{I}.\boldsymbol{T}_3.\dot{\boldsymbol{u}}_3 + \boldsymbol{T}_3.\ddot{\boldsymbol{u}}_3$$

Combine two the vector equation, it can be received

$$\boldsymbol{a}_{O1} + (\varepsilon_1.\boldsymbol{I}.-\omega_1^2.\boldsymbol{E}).\boldsymbol{T}_1.\boldsymbol{u}_1 = \boldsymbol{a}_{O3} + (\varepsilon_3.\boldsymbol{I}.-\omega_3^2.\boldsymbol{E}).\boldsymbol{T}_3.\boldsymbol{u}_3 + 2\omega_3.\boldsymbol{I}.\boldsymbol{T}_3.\dot{\boldsymbol{u}}_3 + \boldsymbol{T}_3.\ddot{\boldsymbol{u}}_3$$

Where $\boldsymbol{a}_{O1} = \boldsymbol{a}_{O3} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$; $\varepsilon_1 = \dot{\omega}_1 = 0$; $\boldsymbol{E} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $\ddot{\boldsymbol{u}}_3 = \begin{Bmatrix} \ddot{s} \\ 0 \end{Bmatrix}$

Expand the vector equation, we get two algebraic equations. Replace the known quantities as θ_1 , θ_3 , ω_1 , ω_3 , ε_1 , s , $\dot{s}$ in the above algebraic equations of acceleration, we continue to solve and determine ε_3 , $\ddot{s}$.

Example 3:(Slider-crank mechanism) [2, 3]

Given: Link 0 = Frame; Link 1 = Crank OA rotating with $\omega = const$ and $\theta = \omega.t$; Link 2 = Connecting rod AB; Link 3 = Slider C; Length of OA = a ; Length of AB = b

Determine: position, sliding velocity, and sliding acceleration of the slider C; position, angular velocity, and angular acceleration of the connecting rod AB. (Fig. 6)

Solution:

This mechanism is a closed chain.

The chain of links: Frame + (revolute joint O) + Crank OA + (revolute joint A) + Connecting rod AB + (revolute joint B) + Slider C

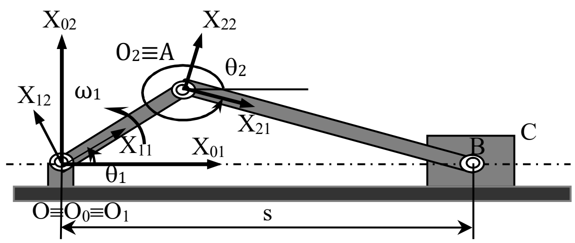

***Fig. 6:** Slider-crank mechanism*

The position of slider C can be determined by equation (1)

$$\boldsymbol{r}_C = \boldsymbol{r}_B = \boldsymbol{r}_O + \boldsymbol{T}_1.\boldsymbol{u}_1 + \boldsymbol{T}_2.\boldsymbol{u}_2$$

Where:

$$\boldsymbol{r}_O = \begin{Bmatrix}0\\0\end{Bmatrix};\ \boldsymbol{T}_1 = \begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix};$$

$$\boldsymbol{T}_2 = \begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix};\ \boldsymbol{u}_1 = \begin{Bmatrix}a\\0\end{Bmatrix};$$

$$\boldsymbol{u}_2 = \begin{Bmatrix}b\\0\end{Bmatrix};\ \theta_1 = \theta = \omega.t;\ \boldsymbol{r}_B = \begin{Bmatrix}s\\0\end{Bmatrix}$$

Replacement and combination of above equations, we get equation systems:

$$\begin{cases}a.\cos\omega.t + b.\cos\theta_2 = s\\ a.\sin\omega.t + b.\sin\theta_2 = 0\end{cases}$$

Solving two algebraic equations, two unknown quantities s and $\theta_2$are found. Those are parameters specifying positions of slider C and connecting rod AB.

The velocity of slider C can be specified by equation (4)

$$\boldsymbol{v}_C = \boldsymbol{v}_B = \boldsymbol{v}_O + \omega_1.\boldsymbol{I}.\boldsymbol{T}_1.\boldsymbol{u}_1 + \omega_2.\boldsymbol{I}.\boldsymbol{T}_2.\boldsymbol{u}_2$$

Where: $\boldsymbol{v}_O = \begin{Bmatrix}0\\0\end{Bmatrix}$; $\boldsymbol{I} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$; $\omega_1 = \omega$; $\boldsymbol{v}_C = \begin{Bmatrix}v\\0\end{Bmatrix}$

Substitution and simplifying, we get

$$\begin{cases}v = -\omega.a.\sin\omega.t - \omega_2.b.\sin\theta_2\\ \omega.a.\cos\omega.t + \omega_2.b.\cos\theta_2 = 0\end{cases}$$

Solving two algebraic equations, two unknown quantities v and ω_2 are found. Those are parameters specifying velocity of slider C and connecting rod AB follow to crank moving.

The acceleration of slider C can be determined by equation (5)

$$\boldsymbol{a}_C = \boldsymbol{a}_B = \boldsymbol{a}_O + (\varepsilon_1.\boldsymbol{I}.-\omega_1^2.\boldsymbol{E}).\boldsymbol{T}_1.\boldsymbol{u}_1 + (\varepsilon_2.\boldsymbol{I}.-\omega_2^2.\boldsymbol{E}).\boldsymbol{T}_2.\boldsymbol{u}_2$$

Where $\boldsymbol{a}_O = \begin{Bmatrix}0\\0\end{Bmatrix}$; $\varepsilon_1 = \dot{\omega}_1 = \dot{\omega} = 0$; $\boldsymbol{E} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$; $\boldsymbol{a}_C = \begin{Bmatrix}a\\0\end{Bmatrix}$

Continue to substitute and simplify the acceleration vector equation, we get two algebraic equations

$$\begin{cases}a = -\omega^2.a.\cos\omega.t - \omega_2^2.b.\cos\theta_2 - \varepsilon_2.b.\sin\theta_2\\ -\omega^2.a.\sin\omega.t + \varepsilon_2.b.\cos\theta_2 - \omega_2^2.b.\sin\theta_2 = 0\end{cases}$$

Solving two algebraic equations, two unknown quantities a andε_2 are found. Those are parameters specifying acceleration of slider C and connecting rod AB.

Example 4: The four-bar mechanism [2, 3]

Given: Link 0 = Frame; Link 1 = Crank OA rotating with $\omega = const$ and $\theta = \omega.t$; Link 2 = Connecting rod AB; Link 3 = Pendulum rod; Length of OA = a; Length of AB = b; Length of BC = c; Distance OC = d; AK = e; Angle KAB = α (Fig. 7)

Determine: position, angular velocity, and angular acceleration of the connecting rod AB and the pendulum rod BC; position, velocity, and acceleration of point K.

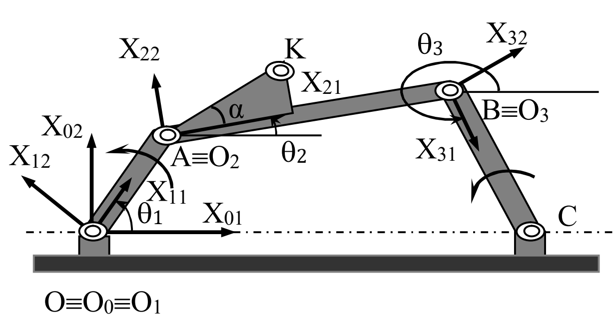

***Fig. 7:** Four-bar mechanism*

Solution:

This mechanism is a closed chain.

The chain of links: Frame + (revolute joint O) + Crank OA + (revolute joint A) + Connecting rod AB + (revolute joint B) + Pendulum rod BC + (revolute joint C)

The position of joint C can be determined by equation (1) and following the chain of links

$$\boldsymbol{r}_C = \boldsymbol{r}_O + \boldsymbol{T}_1.\boldsymbol{u}_1 + \boldsymbol{T}_2.\boldsymbol{u}_2 + \boldsymbol{T}_3.\boldsymbol{u}_3$$

Where:

$$\boldsymbol{r}_O = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix};\ \boldsymbol{T}_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix};$$

$$\boldsymbol{T}_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix};$$

$$\boldsymbol{T}_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{bmatrix};$$

$$\boldsymbol{u}_1 = \begin{Bmatrix} a \\ 0 \end{Bmatrix};\ \boldsymbol{u}_2 = \begin{Bmatrix} b \\ 0 \end{Bmatrix};\ \boldsymbol{u}_3 = \begin{Bmatrix} c \\ 0 \end{Bmatrix};$$

$$\theta_1 = \theta = \omega.t\ ;\ \boldsymbol{r}_C = \begin{Bmatrix} d \\ 0 \end{Bmatrix}$$

Following above process we receive the position equation

$$\begin{cases} a.\cos\omega.t + b.\cos\theta_2 + c.\cos\theta_3 = d \\ a.\sin\omega.t + b.\sin\theta_2 + c.\sin\theta_3 = 0 \end{cases}$$

Solving two nonlinear algebraic equations, two unknown quantities θ_2 and θ_3 are found in a term of t.

The velocity of joint C can be specified by equation (4)

$$\boldsymbol{v}_C = \boldsymbol{v}_O + \omega_1.\boldsymbol{I}.\boldsymbol{T}_1.\boldsymbol{u}_1 + \omega_2.\boldsymbol{I}.\boldsymbol{T}_2.\boldsymbol{u}_2 + \omega_3.\boldsymbol{I}.\boldsymbol{T}_3.\boldsymbol{u}_3$$

Where: $\boldsymbol{v}_O = \boldsymbol{v}_C = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$; $\boldsymbol{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; $\omega_1 = \omega$;

Finally, we get the velocity equation

$$\begin{cases} -\omega.a.\sin\omega.t - \omega_2.b.\sin\theta_2 - \omega_3.c.\sin\theta_3 = 0 \\ \omega.a\cos\omega.t + \omega_2.b.\cos\theta_2 + \omega_3.c.\cos\theta_3 = 0 \end{cases}$$

Solving two linear algebraic equations, two unknown quantities ω_2 and ω_3 are found in a term of t.

The acceleration of joint C can be determined by equation (5)

$$\boldsymbol{a}_C = \boldsymbol{a}_O + (\varepsilon_1.\boldsymbol{I}.-\omega_1^2.\boldsymbol{E}).\boldsymbol{T}_1.\boldsymbol{u}_1 + (\varepsilon_2.\boldsymbol{I}.-\omega_2^2.\boldsymbol{E}).\boldsymbol{T}_2.\boldsymbol{u}_2 + (\varepsilon_3.\boldsymbol{I}.-\omega_3^2.\boldsymbol{E}).\boldsymbol{T}_3.\boldsymbol{u}_3$$

Where $\boldsymbol{a}_O = \boldsymbol{a}_C = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$; $\varepsilon_1 = \dot{\omega}_1 = \dot{\omega} = 0$; $\boldsymbol{E} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Similarly, we get two velocity algebraic equations

$$-\omega^2.a.\cos\omega.t - \omega_2^2.b.\cos\theta_2 - \varepsilon_2.b.\sin\theta_2 - \omega_3^2.c.\cos\theta_3 - \varepsilon_3.c.\sin\theta_3 = 0$$

$$-\omega^2.a.\sin\omega.t + \varepsilon_2.b.\cos\theta_2 - \omega_2^2.b.\sin\theta_2 + \varepsilon_3.c.\cos\theta_3 - \omega_3^2.c.\sin\theta_3 = 0$$

Solving two linear algebraic equations, two unknown quantities ε_2 and ε_3 are found in a term of t.

The position of point K can be specified by equation (1)

$$\boldsymbol{r}_K = \boldsymbol{r}_O + \boldsymbol{T}_1.\boldsymbol{u}_1 + \boldsymbol{T}_2.\boldsymbol{u}_2$$

Where $\boldsymbol{u}_2 = \begin{Bmatrix} e.\cos\alpha \\ e.\sin\alpha \end{Bmatrix}$

Deduce:

$$\begin{cases} x_{K1} = a.\cos\omega.t + e.\cos\alpha.\cos\theta_2 - e.\sin\alpha.\sin\theta_2 \\ x_{K2} = a.\sin\omega.t + e.\cos\alpha.\sin\theta_2 + e.\sin\alpha.\cos\theta_2 \end{cases}$$

The velocity of point K can be determined by equation (4)

$$\boldsymbol{v}_K = \boldsymbol{v}_O + \omega_1.\boldsymbol{I}.\boldsymbol{T}_1.\boldsymbol{u}_1 + \omega_2.\boldsymbol{I}.\boldsymbol{T}_2.\boldsymbol{u}_2$$

Deduce:

$$\begin{cases} v_{C1} = -\omega.a.\sin\omega.t - \omega_2.e.\cos\alpha.\sin\theta_2 - \omega_2.e.\sin\alpha.\cos\theta_2 \\ v_{C2} = \omega.a.\cos\omega.t + \omega_2.e.\cos\alpha.\cos\theta_2 + \omega_2.e.\sin\alpha.\sin\theta_2 \end{cases}$$

The acceleration of point K can be determined by equation (5)

$$\boldsymbol{a}_K = \boldsymbol{a}_O + (\varepsilon_1.\boldsymbol{I}.-\omega_1^2.\boldsymbol{E}).\boldsymbol{T}_1.\boldsymbol{u}_1 + (\varepsilon_2.\boldsymbol{I}.-\omega_2^2.\boldsymbol{E}).\boldsymbol{T}_2.\boldsymbol{u}_2$$

Expand this vector equation we also result in a similar way.

4. RESULTS FROM MATLAB PROGAME

To illustrate support of using computer program in calculating the kinematic quantities, we use the results determined from the example 3 in order to calculate the position, sliding velocity, and sliding acceleration of the slider C; the position, angular velocity, and angular acceleration of the connecting rod AB in a slider-crank mechanism.

Given: the crank with length of *a =45 mm* revolutes with *n = 30 rounds/min = const* (counterclockwise)*;* the length of connecting rod *b = 100 mm*

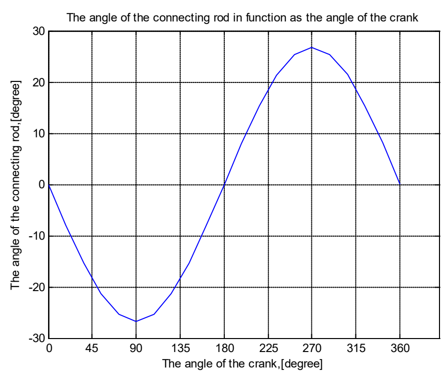

Fig. 8a: *Diagram of angle of the connecting rod in a function as angle of the crank*

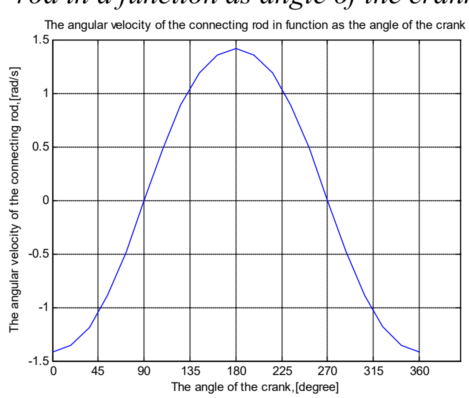

Fig. 8b: *Diagram of angular velocity of the connecting rod in a function as angle of the crank*

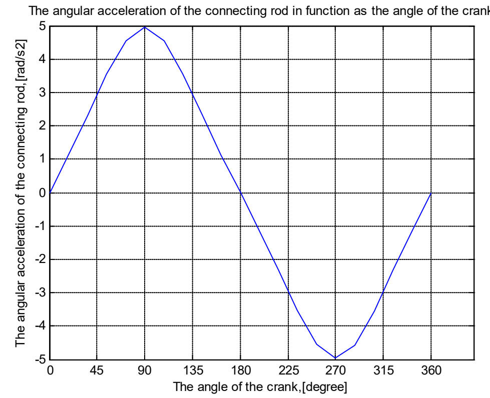

Fig. 8c: *Diagram of angular acceleration of the connecting rod*

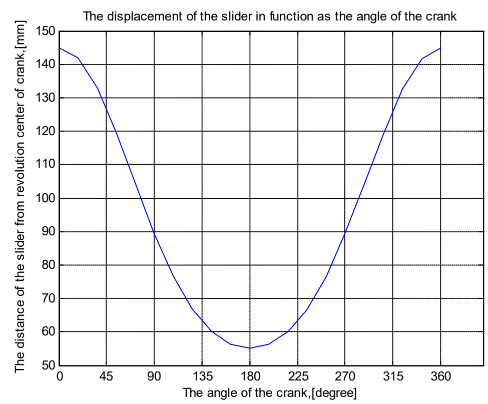

Fig. 8d: *Diagram of displacement of the slider in a function as angle of the crank*

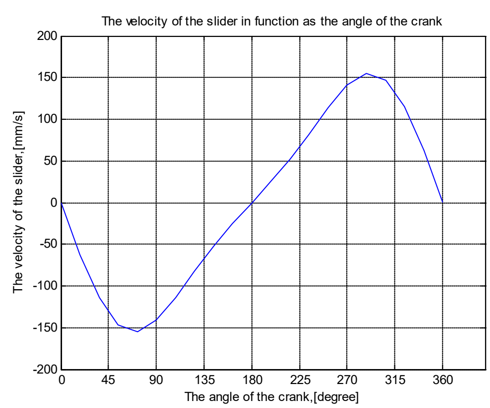

Fig. 8e: *Diagram of sliding velocity of the slider in a function as angle of the crank*

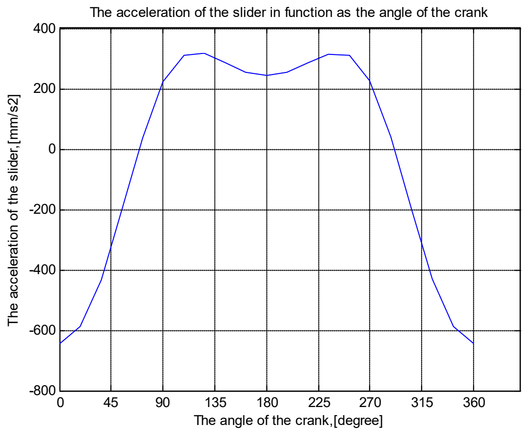

Fig. 8f: *Diagram of sliding acceleration of the slider in a function as angle of the crank*

Fig. 8a-f shows diagrams of position, sliding velocity, and sliding acceleration of the slider; the position, angular velocity, and angular acceleration of the connecting rod in a slider-crank mechanism. (Using Matlab)

5. CONCLUSION

By using the transformation matrix and using basic operations (such as addition, scalar multiplication, and derivative) the position, velocity, and acceleration equation were established for a planar mechanism. From these formulas, analytical kinematics can be applied for some typical planar mechanism. The first advantage of this method is able to calculate position, velocity, and acceleration of links and joints accurately at any position of input link. The second is to develop computation process automatically thanks to support from computer program, such as MatLab, Excel. Some chosen examples illustrate kinematic analytical ability to different planar mechanism. In the future, kinematic designs by graphics can be replaced by analytics in the machinery design subjects in Vietnam universities.

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